B.Sc – I

Waves & Vibrations

Solution of Differential eq^n of a forced oscillator in Vector

The differential eq^n of the oscillator in vector form is

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + Sx = f_0 e^{i\omega t} \quad -(1)$$

Let $x = Ae^{i\omega t}$ –(2) be the sol^n of this eq^n

Differentiating w.r.t 't', we get

$$v = \frac{dx}{dt} = \frac{d}{dt}(Ae^{i\omega t}) = A(i\omega)e^{i\omega t}$$

$$= i\omega(Ae^{i\omega t}) = i\omega x$$

$$\therefore \frac{dx}{dt} = i\omega x$$

Acceleration $\frac{d^2x}{dt^2} = \frac{d}{dt}(i\omega x) = i\omega\frac{d}{dt}(Ae^{i\omega t})$

$$= i^2\omega^2 x = -\omega^2 x \quad (\because i^2 = -1)$$

Put the values of x, $\frac{dx}{dt}$, & $\frac{d^2x}{dt^2}$ in eq^n (1), we get

$$-m\omega^2 A e^{i\omega t} + i\omega r A e^{i\omega t} + SA e^{i\omega t} = f_0 e^{i\omega t}$$

or $Ae^{i\omega t}(-m\omega^2 + i\omega r + S) = f_0 e^{i\omega t}$

or $A = \frac{f_0 \cancel{e^{i\omega t}}}{-m\omega^2 + i\omega r + S} = \frac{f_0}{i\omega r + (S - m\omega^2)}$

$$= \frac{f_0}{\omega\left[ir + \left(\frac{S}{\omega} - m\omega\right)\right]}$$

Multiply & divide by $-i$

$$A = \frac{-if_0}{\omega\left[-i^2 r + (-i)\left(\frac{S}{\omega} - m\omega\right)\right]} = \frac{-if_0}{\omega\left[r - i\left(\frac{S}{\omega} - m\omega\right)\right]}$$

$$A = \frac{-iF_0}{\omega\left[r+i\left(m\omega - \frac{s}{\omega}\right)\right]}$$

$\therefore$ eq^n (2) becomes

$$x = \frac{-iF_0}{\omega\left[r+i\left(m\omega - \frac{s}{\omega}\right)\right]} e^{i\omega t}$$

Put $r + i\left(m\omega - \frac{s}{\omega}\right) = Z_m$ (mechanical impedance of the oscillator)

$$x = \frac{-iF_0 \, e^{i\omega t}}{\omega Z_m} \quad \text{--- (3)}$$

measure of opposition to the current flow by the material

(1) Resistive part : 'r' is the mechanical resistance
(2) Reactive " : $\left(m\omega - \frac{s}{\omega}\right)$ " " reactance

resistance offered to the a.c currents by inductors & capacitors only.

Put $r = Z_0 \cos\phi$ & $m\omega - \frac{s}{\omega} = Z_0 \sin\phi$

then $Z_m = Z_0(\cos\phi + i\sin\phi) = Z_0 e^{i\phi}$ --- (4)

{Resistance - measures the opposition to a flow of current
Reactance - " " " to a change in current}

Put (4) in (3)

$$x = \frac{-iF_0 \, e^{i\omega t}}{\omega Z_0 e^{i\phi}} = \frac{-iF_0 \, e^{i(\omega t - \phi)}}{\omega Z_0}$$

This eq^n represents the steady state behavior of forced oscillator.

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Displacement with Driving force frequency (ω)

The displacement of the forced oscillator is

$$x = -\frac{i F_0}{\omega Z_0} e^{i(\omega t - \phi)} = -\frac{i F_0}{\omega Z_0}\left[\cos(\omega t-\phi) + i\sin(\omega t-\phi)\right]$$

$$= -\frac{i F_0}{\omega Z_0}\cos(\omega t-\phi) + \underbrace{\frac{F_0}{\omega Z_0}\sin(\omega t-\phi)}_{\text{real part of displacement}}$$

Displacement amplitude $= \frac{F_0}{\omega Z_0}$

① Amplitude at low driving freq. (ω)

When driving freq. is low, $\omega \to 0$

Displacement amplitude (A_L) $= \frac{F_0}{\omega Z_0} = \frac{F_0}{\omega\left[r^2 + \left(m\omega - \frac{s}{\omega}\right)^2\right]^{1/2}}$

$$\left[\because Z_0 = \left[r^2 + \left(m\omega - \frac{s}{\omega}\right)^2\right]^{1/2}\right]$$

Since $\omega \to 0$

$$Z_0 = \left[r^2 + \frac{s^2}{\omega^2}\right]^{1/2}$$

If r is negligibly small, then

$$Z_0 \simeq \frac{s}{\omega}$$

$$\therefore A_L \simeq \frac{F_0}{\omega Z_0} = \frac{F_0}{\omega \times \frac{s}{\omega}} = \frac{F_0}{s}$$

This shows that the amplitude of the forced oscillator at low driving frequencies depends upon the driving force & the stiffness constant s & independent of the freq. of the applied force.

At high driving frequencies

$\omega \to \infty$

$$A_H = \frac{F_0}{\omega\left[r^2 + \left(m\omega - \frac{S}{\omega}\right)^2\right]^{1/2}}$$

$$Z_0 \left[r^2 + \left(m\omega - \frac{S}{\omega}\right)^2\right]^{1/2}$$

As $\omega \to \infty$, $m\omega \to \infty$ & $\frac{S}{\omega} \to 0$

$$Z_0 \cong m\omega \to \infty$$

$$A_H = \frac{F_0}{m\omega} = \frac{F_0}{\infty} = 0$$

8. Freq. at which displacement resonance occurs

Disp. Amplitude $A = \frac{F_0}{\omega Z_0}$

Resonance will occur when ωZ_0 is min^m. It will be min^m when its differential coeff. w.r.t ω is zero

i.e $\frac{d}{d\omega}(\omega Z_0) = 0$

$$\frac{d}{d\omega}\,\omega\left[r^2 + \left(m\omega - \frac{S}{\omega}\right)^2\right]^{1/2} = 0 \qquad \frac{d}{d\omega}\left[r^2\omega^2 + (m\omega^2 - S)^2\right]^{1/2} = 0$$

$$\frac{1}{2}\left[r^2\omega^2 + (m\omega^2 - S)^2\right]^{-1/2}\left[2\omega r^2 + 4m\omega(m\omega^2 - S)\right] = 0$$

$$\frac{2\omega r^2 + 4m\omega(m\omega^2 - S)}{2\left[r^2\omega^2 + (m\omega^2 - S)^2\right]^{1/2}} = 0$$

$$\left\{\begin{aligned}&\frac{d(m^2\omega^4 + S^2 - 2m\omega^2 S)}{d\omega}\\ &= 4m^2\omega^3 - 2\times 2m\omega S\\ &\;\; 4m\omega(m\omega^2 - S)\end{aligned}\right\}$$

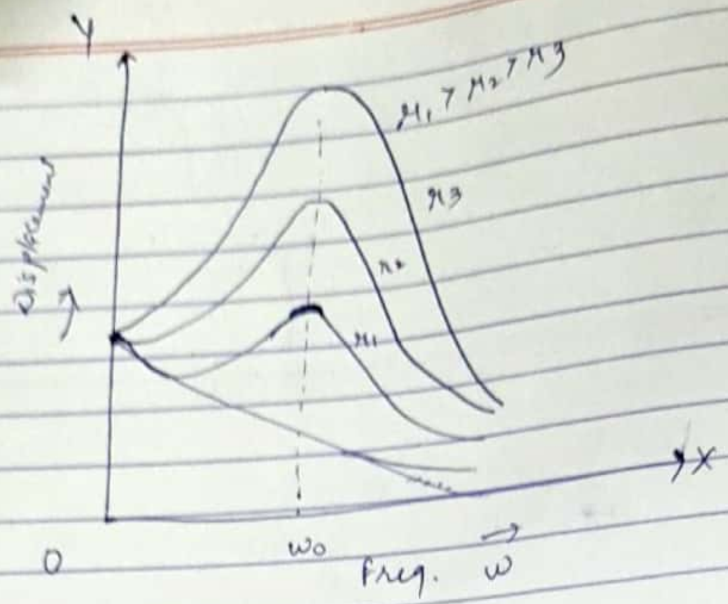
or $2\omega\left[r^2 + 2m(m\omega^2 - S)\right] = 0$

So either $\omega = 0$, or $r^2 + 2m(m\omega^2 - S) = 0$

or $m\omega^2 - S = -\frac{r^2}{2m} \Rightarrow \omega^2 = \frac{S}{m} - \frac{r^2}{2m^2}$

$$\text{or } \omega = \left(\omega_0^2 - \frac{r^2}{2m^2}\right)^{1/2} = \omega' \quad \left\{\text{where } \omega_0 = \sqrt{S/m}\right\}$$

As r decreases, the resonance freq. shifts towards the resonance freq. ω_0



Velocity Variation with D.F.F

The velocity of force oscillator is

$$v = \frac{F_0}{Z_0} \cos(\omega t - \phi)$$

The velocity amplitude $A_v = \frac{F_0}{Z_0}$

$$A_v = \frac{F_0}{\left[r^2 + \left(m\omega - \frac{S}{\omega}\right)^2\right]^{1/2}} \quad -(1)$$

(a) At low freq.

$\omega \to 0$

$m\omega \to 0$ & $\frac{S}{\omega} \to \infty$

$$Z_0 = \left[r^2 + \frac{S^2}{\omega^2}\right]^{1/2} \simeq \frac{S}{\omega}$$ as r is very small.

$\therefore$ eq^n (1) becomes

$$A_v = \frac{F_0}{\frac{S}{\omega}} = F_0 \frac{\omega}{S}$$

Since $\omega \to 0$ $\therefore A_v \to 0$

(b) At high driving freq. $\omega \to \infty$

$m\omega \to \infty$ & $\frac{S}{\omega} \to 0$

$\therefore Z_0 \cong m\omega$ ($\because$ r is very small)

$\therefore A_H = \frac{F_0}{m\omega}$

Since, $\omega \to \infty$, $\therefore A_H \to 0$

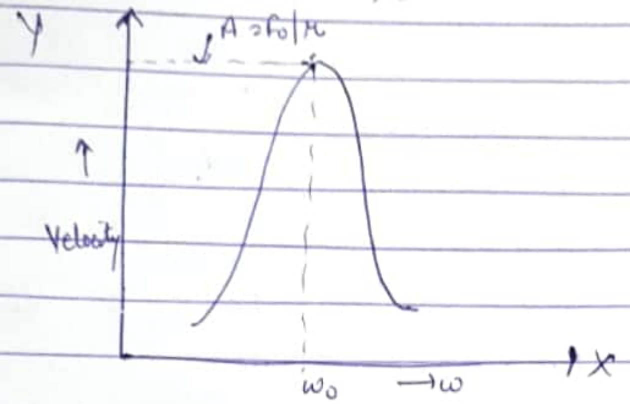
(C) At resonant frequency

In this case $m\omega = \frac{S}{\omega}$

The impedance has min^m value

$Z_0 \simeq r$

$\therefore A_V = \frac{F_0}{r}$



Variation of Phase with DFF.

The phase diff. b/w the velocity & driving force is $-\phi$. The velocity lags behind the " " by an angle ϕ such that

$$\tan\phi = \frac{m\omega - S/\omega}{r}$$

$\left[\because \tan\phi = \frac{Z_0 \sin\phi}{Z_0 \cos\phi} \quad Z_0 \sin\phi = m\omega - \frac{S}{m}\right]$

$Z_0 \cos\phi = r$

(a) At low freq.

$\omega \to 0$

$m\omega \to 0$ & $\frac{S}{\omega} \to \infty$

$$\tan\phi = \frac{0 - \infty}{r} = -\infty = -\tan 90^\circ = \tan(-90^\circ)$$

$$\phi = -90^\circ = -\frac{\pi}{2} \text{ radians}$$

$\therefore$ The p.d b/w velocity & the force $= -\phi = -(-90^\circ) = 90^\circ$

Hence, the resulting velocity leads the driving force by $\frac{\pi}{2}$ radians.

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(b) At high frequencies

$\omega \to \infty$

$$\tan\phi = \frac{m\omega - \frac{S}{\omega}}{r} \simeq \infty = \tan 90^\circ$$

$$\phi = 90^\circ = \frac{\pi}{2} \text{ radians}$$

∴ The P.D b/w velocity & Driving force $= (-\phi) = -90^\circ = \frac{\pi}{2}$

The resulting velocity lags behind the D.F by $\frac{\pi}{2}$ radians

(c) At resonance (when $\omega = \omega_0$)

$$m\omega = \frac{S}{\omega}$$

$$\tan\phi = \frac{m\omega - \frac{S}{\omega}}{r} = 0 = \tan 0 \text{ or } \phi = 0$$

∴ The P.D b/w velocity & force is zero or they are in phase with each other.