CALCULUS – II

DOT PRODUCT

Definition

If $\mathbf{a} = <a_1,a_2,a_3>$ and $\mathbf{b} = <b_1,b_2,b_3>$, then the **dot product** of **a** and **b** is a scalar *c* given by

$$c = \mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Another name for dot product is **scalar product**.

Example

If **a** = (1, -3, 2) and **b** = (4, 5, -8), find the dot product of **a** and **b.**

Solution

$$c = \mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

$$= (1)x(1) + (-3)x(5) + (2)x(-8)$$

$$= -27$$

PROPERTIES OF THE DOT PRODUCT

If **a**, **b**, and **c** are 3-D vectors, then

1. $\mathbf{a}.\mathbf{a} = |a|^2$
2. $\mathbf{a}.\mathbf{b} = \mathbf{b}.\mathbf{a}$
3. $\mathbf{a}.(\mathbf{b} + \mathbf{c}) = \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c}$
4. $(\mathbf{c}.\mathbf{a}).\mathbf{b} = \mathbf{c}.(\mathbf{a}.\mathbf{b}) = \mathbf{a}.(\mathbf{c}.\mathbf{b})$
5. $0.\mathbf{a} = 0$

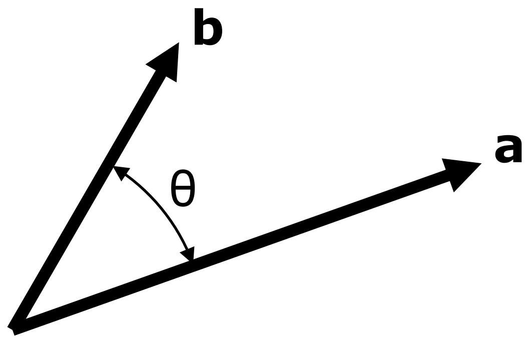
Geometric Interpretation of Dot Product

If θ is the angle between the nonzero vectors **a** and **b**, then

$$\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

or

$$\cos \boldsymbol{\theta} = \mathbf{a}.\mathbf{b} \,/\, |\mathbf{a}||\mathbf{b}|$$



Note
Two non-zero vectors **a** and **b** are orthogonal if and only if

$$\mathbf{a}\cdot\mathbf{b} = 0$$

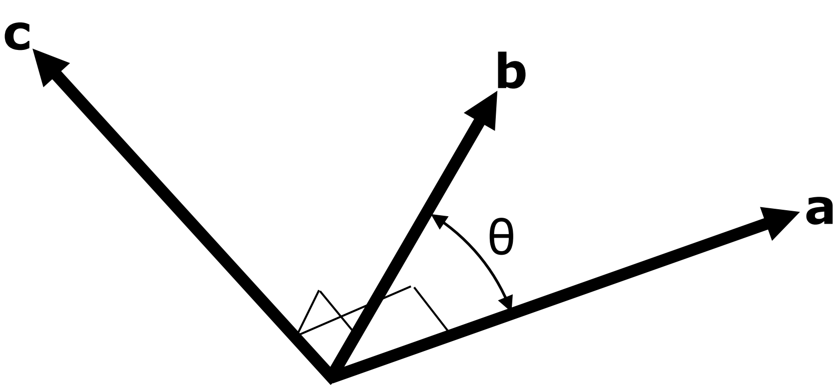
CROSS PRODUCT OF TWO VECTORS

If **a** = a_1**i** + a_2**j** + a_3**k** and **b** = b_1**i**+ b_2**j** + b_3**k**, then the **cross product** of **a** and **b** is written as **a x b.**

The cross product **a x b** is a vector **c** such that

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= (a_2 b_3 - b_2 a_3)\mathbf{i} - (a_1 b_3 - b_1 a_3)\mathbf{j} - (a_1 b_2 - b_1 a_2)\mathbf{k}$$



Note
The vector **c** is orthogonal to both a and b.

Example:

Let **A** = (1, -3, 2) and **B** = (4, 5, -8), then

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 2 \\ 4 & 5 & -8 \end{vmatrix}$$

$$= ((-3)(-8) - (2)(50))\mathbf{i} - ((1)(-8) - (4)(2))\mathbf{j} - ((1)(5) - (-3)(4))\mathbf{k}$$

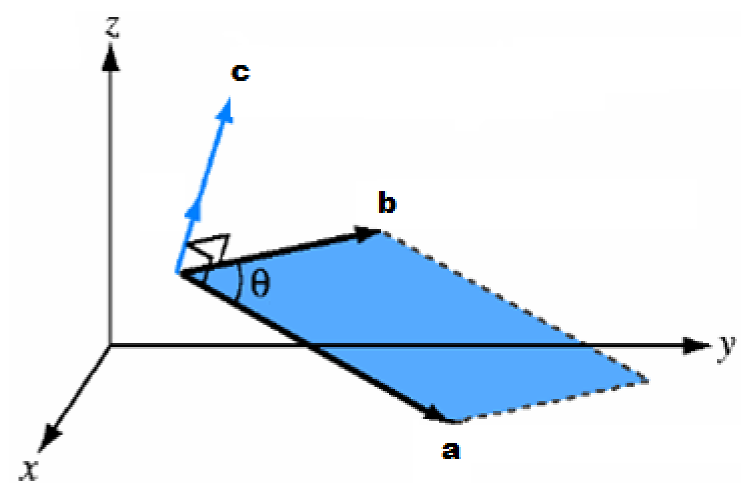
$$= 14\mathbf{i} + 16\mathbf{j} + 17\mathbf{k}$$

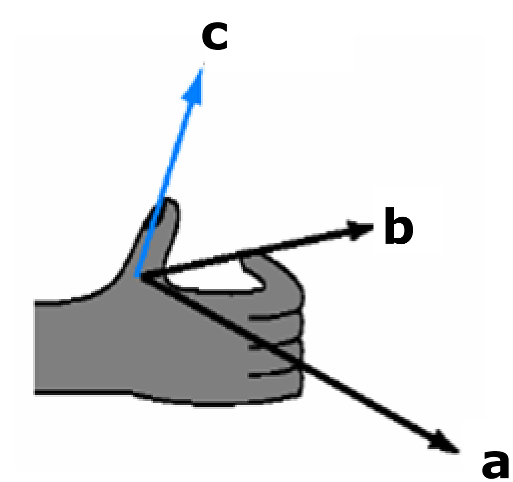
GEOMETRIC INTEPRETATION

If θ is the angle between the nonzero vectors **a** and **b**, then the cross product of **a** and **b** is a vector **c** whose magnitude is given by the expression

$$c = \mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta$$

and whose direction is given by the right hand rule.





Note
Two nonzero vectors **a** and **b** are parallel if and only if **a** x **b** = 0

CROSS PRODUCT OF BASIS VECTORS

i x j = k ; j x k = I

k x i = j ; j x i = -k

k x j = -i ; i x k = -j

i x j ≠ j x i

THEOREM

If **a** an**d b** are vectors and **c** is a scalar, then

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

$$(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$$

Applications

- A simple cross product

$$(5\hat{i}-\hat{j}+2\hat{k})\times(2\hat{i}+3\hat{j}-\hat{k})$$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 5 & -1 & 2\\ 2 & 3 & -1\end{vmatrix}$$

$$=[(-1.-1)-3.2]\hat{i} \rightarrow -5$$

$$+[2.2-(5.-1)]\hat{j} \rightarrow 9$$

$$+[5.3-(2.-1)]\hat{k} \rightarrow 17$$

An Example

$$(5\hat{i}-\hat{j}+2\hat{k})\times(2\hat{i}+3\hat{j}-\hat{k})$$

$$=-5\hat{i}+9\hat{j}+17\hat{k}$$

C $(\vec{a}\times\vec{b})$ is perpendicular to $\vec{a}\ \&\ \vec{b}$

H

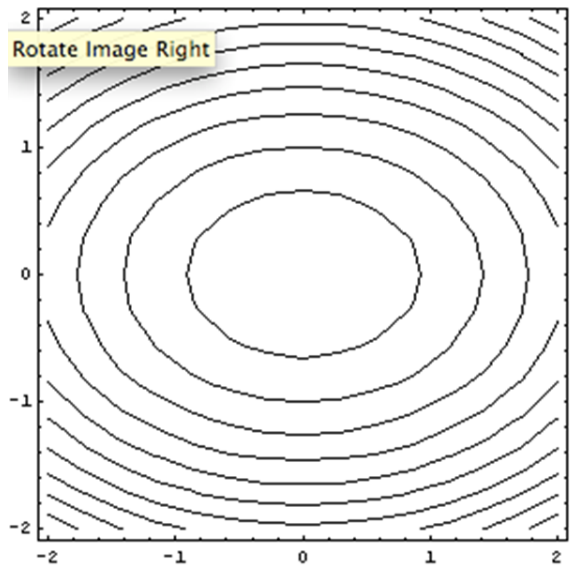
E $(5\hat{i}-\hat{j}+2\hat{k}).(-5\hat{i}+9\hat{j}+17\hat{k})$

C $=-5.5-1.9+2.17$

K $=0$

The basics

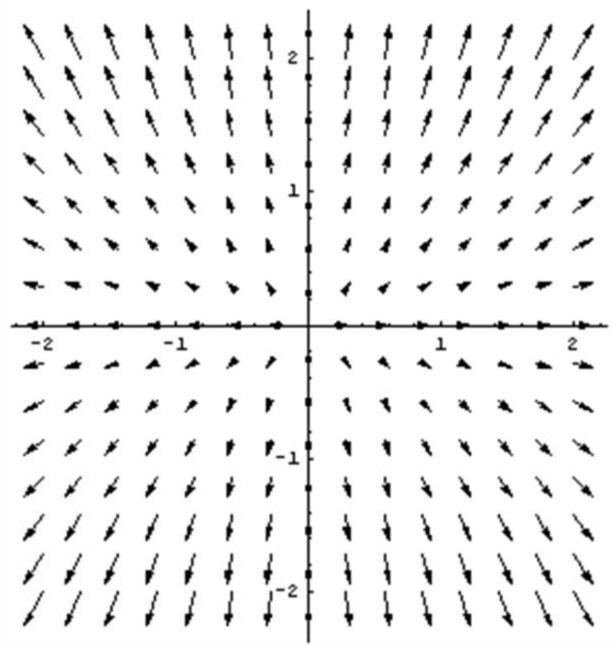
A scalar field associates a single number (or scalar) to every point in space.



Question: Examples for a scalar field?

Answer: Temperature, topography and pressure.

A vector field associates a vector to every point in space.



Question: Examples for a vector field?

Answer: Speed, magnetic field and forces, such as the gravitational force.

VECTOR AND SCALAR FUNCTIONS

A vector valued function **A**(t) is a rule that associates with each real number t a vector **A**(t).

$$\mathbf{A}(t) = A_1(t)\mathbf{i} + A_2(t)\mathbf{j} + A_3(t)\mathbf{k}$$

For example, $f(t) = t^3 - 2t + 4$ is a scalar function of a single variable t, while $\mathbf{A}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$ is a vector function of t.

VECTOR DIFFERENTIATION

A vector function $\mathbf{A}(t)$ is differentiable at a point t if

$$\mathbf{A}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{A}(t + \Delta t) - \mathbf{A}(t)}{\Delta t}$$

exists, and $\mathbf{A}'(t)$ is called the derivative of $\mathbf{A}(t)$, written as

$$\mathbf{A}'(t) = A_1'(t)\mathbf{i} + A_2'(t)\mathbf{j} + A_3'(t)\mathbf{k}$$

Calculate the derivative of each component!

Example:

Let $\mathbf{A}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$. Find the derivative of $\mathbf{A}(t)$.

Solution:

$$\mathbf{A}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}$$

RULES OF VECTOR DIFFERENTIATION

$$\frac{d\mathbf{A}}{dt} = 0, \text{ if } \mathbf{A} = \text{constant.}$$

$$\frac{d}{dt}(p\mathbf{A}) = p\frac{d\mathbf{A}}{dt} + \frac{dp}{dt}\mathbf{A}$$

$$\frac{d}{dt}(\mathbf{A} + \mathbf{B}) = \frac{d\mathbf{A}}{dt} + \frac{d\mathbf{B}}{dt}$$

$$\frac{d}{dt}(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot \frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt} \cdot \mathbf{B}$$

$$\frac{d}{dt}\mathbf{A}(u) = \frac{d\mathbf{A}}{du}\frac{du}{dt}$$

$$\frac{d}{dt}(\mathbf{A} \times \mathbf{B}) = \mathbf{A} \times \frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt} \times \mathbf{B}$$

VECTOR INTEGRATION

Let $\mathbf{A}(t) = A_1(t)\mathbf{i} + A_2(t)\mathbf{j} + A_3(t)\mathbf{k}$ and suppose that the component functions $A_1(t)$, $A_2(t)$ and $A_3(t)$ are integrable. Then the indefinite integral of $\mathbf{A}(t)$ is defined by

$$\int \mathbf{A}(t)dt = \mathbf{i}\int A_1(t)dt + \mathbf{j}\int A_2(t)dt + \mathbf{k}\int A_3(t)dt$$

Calculate the integral of each component!

If $A_1(t)$, $A_2(t)$ and $A_3(t)$ are integrable over the interval $[t_1, t_2]$, then the definite integral of $\mathbf{A}(t)$ is defined by

$$\int_{t_1}^{t_2} \mathbf{A}(t)dt = \mathbf{i}\int_{t_1}^{t_2} A_1(t)dt + \mathbf{j}\int_{t_1}^{t_2} A_2(t)dt + \mathbf{k}\int_{t_1}^{t_2} A_3(t)dt$$

Example

Let $\mathbf{A}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$. Find $\int_0^{2\pi} \mathbf{A}(t)dt$

Solution:

$$\int_0^{2\pi} \mathbf{A}(t)dt = \mathbf{i}\int_0^{2\pi} A_1(t)dt + \mathbf{j}\int_0^{2\pi} A_2(t)dt + \mathbf{k}\int_0^{2\pi} A_3(t)dt$$

$$= \mathbf{i}\int_0^{2\pi} \cos t dt + \mathbf{j}\int_0^{2\pi} \sin t dt + \mathbf{k}\int_0^{2\pi} t dt$$

$$= \mathbf{i}\sin t\Big|_0^{2\pi} - \mathbf{j}\cos t\Big|_0^{2\pi} + \mathbf{k}\frac{1}{2}t^2\Big|_0^{2\pi}$$

$$= 2\pi^2\mathbf{k}$$

SCALAR FIELD

If every point in a region of space is assigned a scalar value obtained from a scalar function $f(x, y, z)$, then a **scalar field** $f(x, y, z)$ is defined in the region, such as the pressure in atmosphere and mass density within the earth, etc.

Partial Derivatives

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, \ldots) - f(x, y, \ldots)}{\Delta x}$$

Mixed second partials

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \frac{\partial f}{\partial x}$$

Example

Let $f = x^2 + 2y^2$. Calculate $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$

Solution:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 2y^2)$$
$$= 2x$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + 2y^2)$$
$$= 4y$$

GRADIENT

Del operator

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

Gradient

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Gradient characterizes maximum increase. If at a point P the gradient of f is not the zero vector, it represents the direction of maximum space rate of increase in f at P.

Example

Given potential function $V = x^2y + xy^2 + xz^2$, (a) find the gradient of V, and (b) evaluate it at (1, -1, 3).

Solution:

(a)

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{i} + \frac{\partial V}{\partial y}\mathbf{j} + \frac{\partial V}{\partial z}\mathbf{k}$$

$$= (2xy + y^2 + z^2)\mathbf{i} + (x^2 + 2xy)\mathbf{j} + 2xz\mathbf{k}$$

(b)

$$\nabla V\Big|_{(1,-1,3)} = (-2+1+9)\mathbf{i} + (1-2)\mathbf{j} + 6\mathbf{k}$$

$$= 8\mathbf{i} - \mathbf{j} + 6\mathbf{k}$$

$$\hat{\mathbf{a}} = \frac{8\mathbf{i} - \mathbf{j} + 6\mathbf{k}}{\sqrt{8^2 + (-1)^2 + 6^2}} = \frac{1}{\sqrt{101}}(8\mathbf{i} - \mathbf{j} + 6\mathbf{k})$$

Direction of maximum increase

VECTOR FIELD

Electric field: $\mathbf{E} = \mathbf{E}(x, y, z)$,

Magnetic field : $\mathbf{H} = \mathbf{H}(x, y, z)$

If every point in a region of space is assigned a vector value obtained from a vector function $\mathbf{A}(x, y, z)$, then a **vector field** $\mathbf{A}(x, y, z)$ is defined in the region.

$$\frac{\partial \mathbf{A}}{\partial t_l} = \frac{\partial A_1}{\partial t_l}\mathbf{i} + \frac{\partial A_2}{\partial t_l}\mathbf{j} + \frac{\partial A_3}{\partial t_l}\mathbf{k}$$

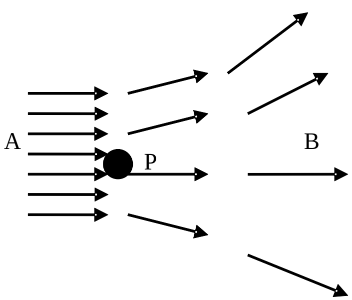
DIVERGENCE OF A VECTOR FIELD

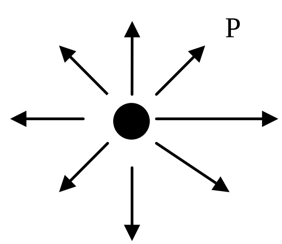
The divergence of a vector field **A** at a point is defined as the net outward flux of **A** per unit volume as the volume about the point tends to zero:

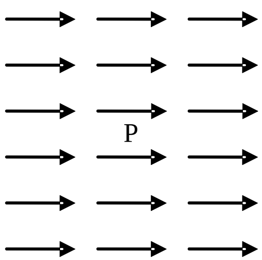
$$\mathbf{div}\,\mathbf{A} = \lim \frac{\oint_S \mathbf{A}.d\mathbf{s}}{\Delta v}$$

It indicates the presence of a source (or sink)! — term the source as **flow source**. And $\mathrm{div}\,\mathbf{A}$ is a measure of the strength of the flow source.

Representing field variations graphically by directed field lines - flux lines







DIVERGENCE OF A VECTOR FIELD

In rectangular coordinate, the divergence of **A** can be calculated as

$$div\,\mathbf{A} = \nabla \cdot A = \left(\frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}} \right) \cdot \left(A_1\hat{\mathbf{i}} + A_2\hat{\mathbf{j}} + A_3\hat{\mathbf{k}} \right)$$

$$= \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

For instance, if $\mathbf{A} = 3xz\mathbf{i} + 2xy\mathbf{j} - yz^2\mathbf{k}$, then

$$\text{div}\,\mathbf{A} = 3z + 2x - 2yz$$

At $(1, 2, 2)$, $\text{div}\,\mathbf{A} = 0$; at $(1, 1, 2)$, $\text{div}\,\mathbf{A} = 4$, there is a source; at $(1, 3, 1)$, $\text{div}\,\mathbf{A} = -1$, there is a sink.

CURL OF A VECTOR FIELD

The curl of a vector field **A** is a vector whose magnitude is the maximum net circulation of **A** per unit area as the area tends to zero and whose direction is the normal direction of the area.

$$curl\,\mathbf{A} = \lim_{\Delta s \to 0} \frac{\oint_L A \cdot dl}{\Delta s}$$

It is an indication of a **vortex source**, which causes a circulation of a vector field around it.

Water whirling down a sink drain is an example of a vortex sink causing a circulation of fluid velocity.

If **A** is electric field intensity, then the circulation will be an electromotive force around the closed path.

Water vortex

CURL OF A VECTOR FIELD

In rectangular coordinate, $\text{curl}\ \mathbf{A}$ can be calculated as

$$\text{curl}\ \mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) \mathbf{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \mathbf{k}$$

Example:

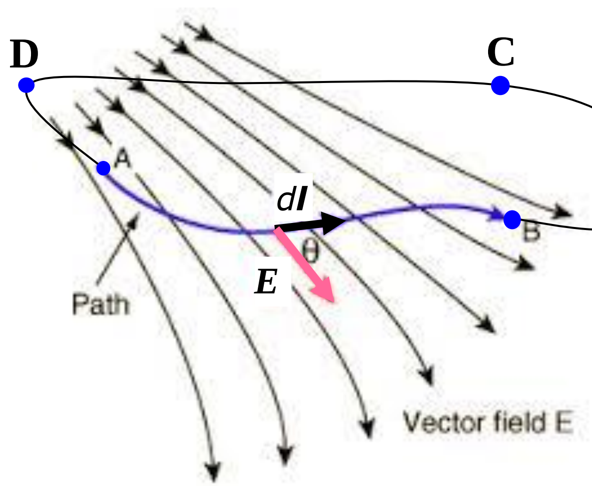
If $\mathbf{A} = yz\mathbf{i} + 3zx\mathbf{j} + z\mathbf{k}$, then

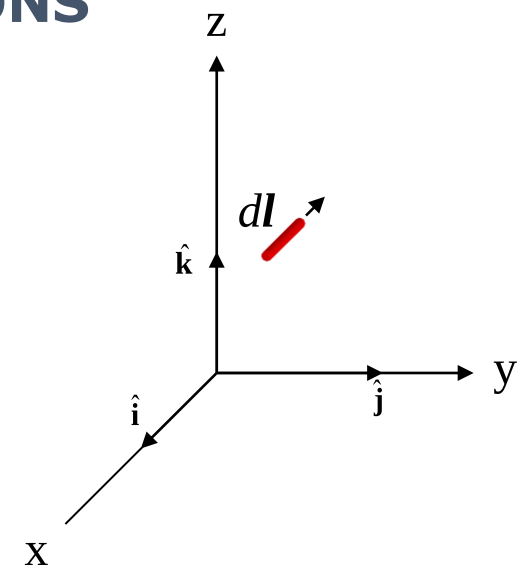
$$\text{curl } \mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & 3zx & z \end{vmatrix}$$

$$= \left(\frac{\partial z}{\partial y} - \frac{\partial (3zx)}{\partial z} \right) \mathbf{i} + \left(\frac{\partial (yz)}{\partial z} - \frac{\partial z}{\partial x} \right) \mathbf{j} + \left(\frac{\partial (3zx)}{\partial x} - \frac{\partial (yz)}{\partial y} \right) \mathbf{k}$$

$$= -3x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}$$

LINE INTEGRAL OF VECTOR FUNCTIONS





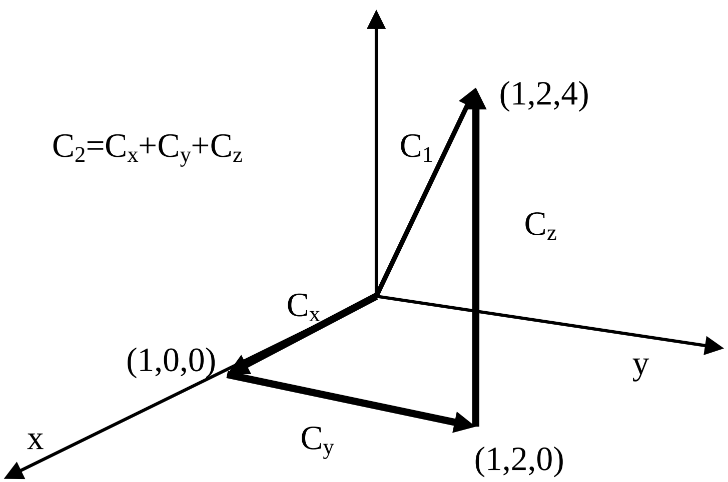
$$\int_{Path\,\mathbf{L}} \mathbf{E}\cdot d\mathbf{l} = \int_A^B E\cos\theta dl \quad ; \quad d\mathbf{l} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$$

For a closed loop, i.e. ABCA,

$$\oint_{Path\,\mathbf{L}} \mathbf{E}\cdot d\mathbf{l} = \text{circulation of } \mathbf{E} \text{ around } L$$

Example

For $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$, calculate the circulation of $\mathbf{F}$ along the two paths as shown below.



Solution:

$$d\mathbf{l} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$$

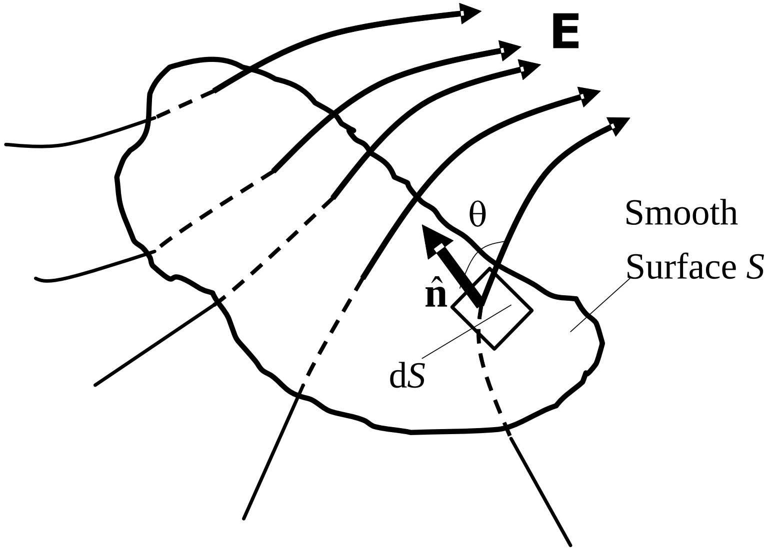
$$\mathbf{F} \bullet d\mathbf{l} = ydx - xdy$$

Along path C_2

$$\int_{C_2} \mathbf{F} \bullet d\mathbf{l} = \int_{C_x} \mathbf{F} \bullet d\mathbf{l} + \int_{C_y} \mathbf{F} \bullet d\mathbf{l} + \int_{C_z} \mathbf{F} \bullet d\mathbf{l}$$

$$\int_{C_2} \mathbf{F} \bullet d\mathbf{l} = \int_0^1 ydx\Big|_{y=0} + \int_0^2 -xdy\Big|_{x=1} + \int_0^4 0dz = -2$$

SURFACE INTEGRAL



Surface integral or the flux of $\mathbf{E}$ across the surface S is

$$\psi = \oint_S \mathbf{E}.\hat{\mathbf{n}} dS = \oint_S E\cos\theta dS$$

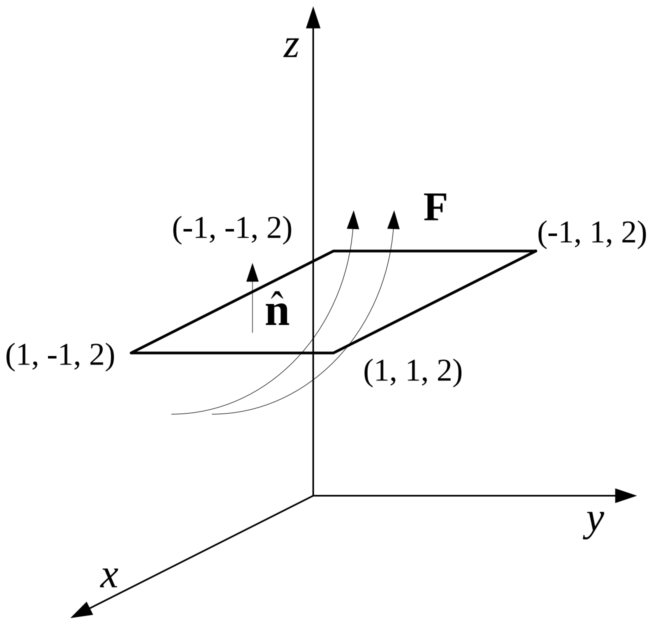
$\hat{\mathbf{n}}$ is the outward unit vector normal to the surface.

For closed surface,

$$\psi = \oint_S \mathbf{E}.\hat{\mathbf{n}} dS = \text{ net outward flux of } \mathbf{E}.$$

Example

If $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + (z^2 - 1)\mathbf{k}$, calculate the flux of $\mathbf{F}$ across the surface shown in the figure.



Solution:

$$\psi = \int_S \mathbf{F} \bullet \hat{\mathbf{n}} dS$$

$$= \int_{-1}^{1}\int_{-1}^{1} (x\mathbf{i} + y\mathbf{j} + (z^2 - 1)\mathbf{k})\Big|_{z=2} \bullet \mathbf{k}\, dxdy$$

$$= \int_{-1}^{1}\int_{-1}^{1} (z^2 - 1)\Big|_{z=2}\, dxdy$$

$$= \int_{-1}^{1}\int_{-1}^{1} 3dxdy$$

$$= 12$$

Example:

Let $\mathbf{F} = \mathbf{2}xz\mathbf{i} - x\mathbf{j} + y^2\mathbf{k}$. Evaluate $\iiint_V \mathbf{F}\,dV$

where V is the region bounded by the surface $x = 0$, $x = 2$, $y = 0$, $y = 6$, $z = 0$, $z = 4$.

Solution:

$$\iiint_V \mathbf{F}\,dV = \int_{x=0}^{x=2}\int_{y=0}^{y=6}\int_{z=0}^{z=4}(2xz\mathbf{i} - x\mathbf{j} + y^2\mathbf{k})\,dzdydx$$

$$= \mathbf{i}\int_{x=0}^{x=2}\int_{y=0}^{y=6}\int_{z=0}^{z=4} 2xz\,dzdydx + \mathbf{j}\int_{x=0}^{x=2}\int_{y=0}^{y=6}\int_{z=0}^{z=4}(-x)\,dzdydx$$

$$+\mathbf{k}\int_{x=0}^{x=2}\int_{y=0}^{y=6}\int_{z=0}^{z=4} y^2\,dzdydx$$

$$= 192\mathbf{i} - 48\mathbf{j} + 576\mathbf{k}$$

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