CALCULUS – II

SCALARS

Scalars are quantities having only a magnitude.

- Length, mass, temperature etc.

VECTORS

1. The term **vector** is used by scientists to indicate a quantity (such as displacement or velocity or force) that has both magnitude and direction.

2. A vector is often represented by an arrow or a directed line segment.

3. We denote a vector by printing a letter in boldface (**v**) or by putting an arrow above the letter ($\vec{v}$).

VECTOR REPRESENTATION

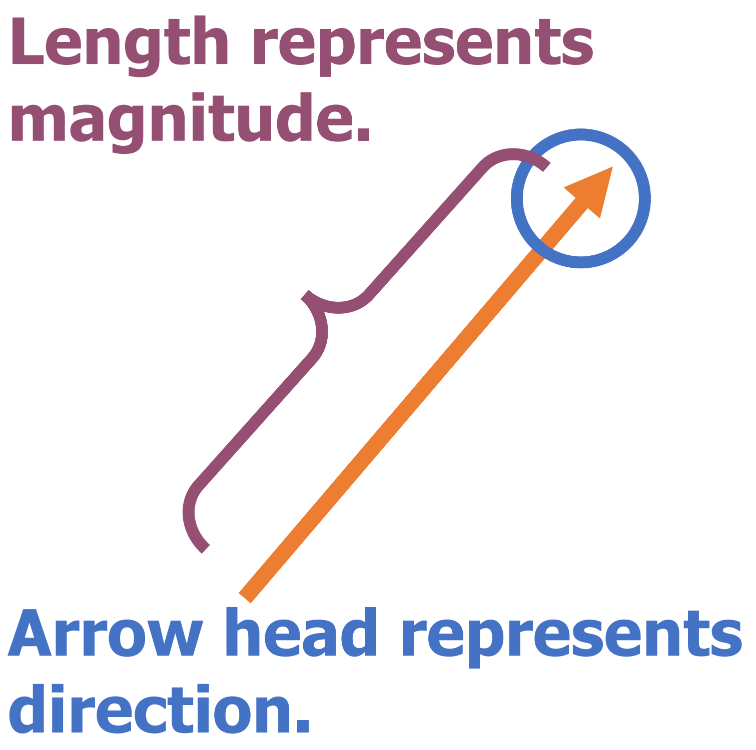


Figure 1. Vector representation

EXAMPLE: FORCE VECTOR

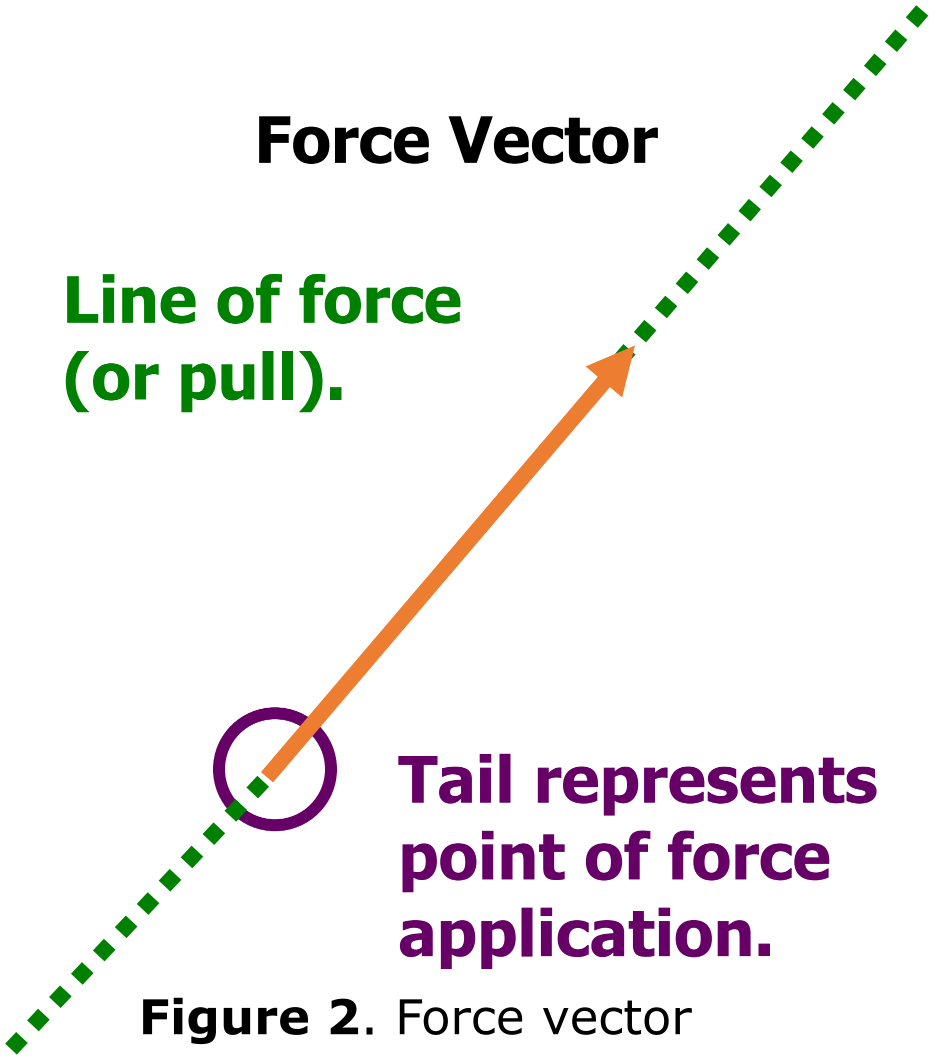


Figure 2. Force vector

COORDINATE SYSTEMS

CARTESIAN COORDINATE SYSTEM

- has three orthogonal axes
- axes are labelled as x, y, and z.
- O is origin.

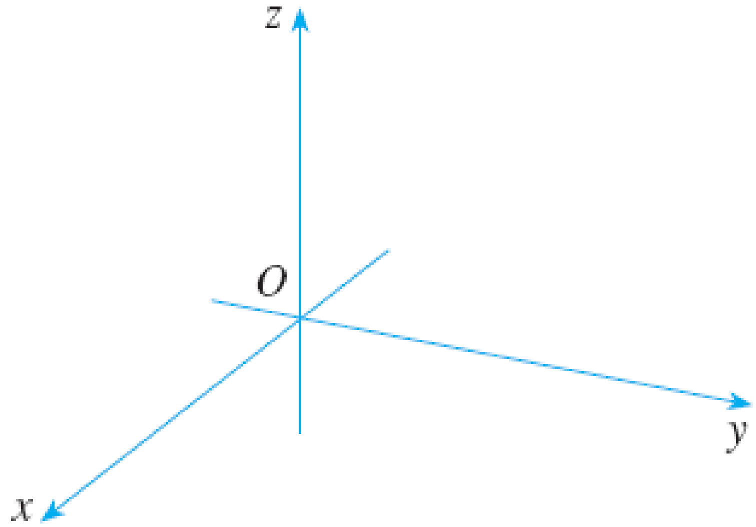
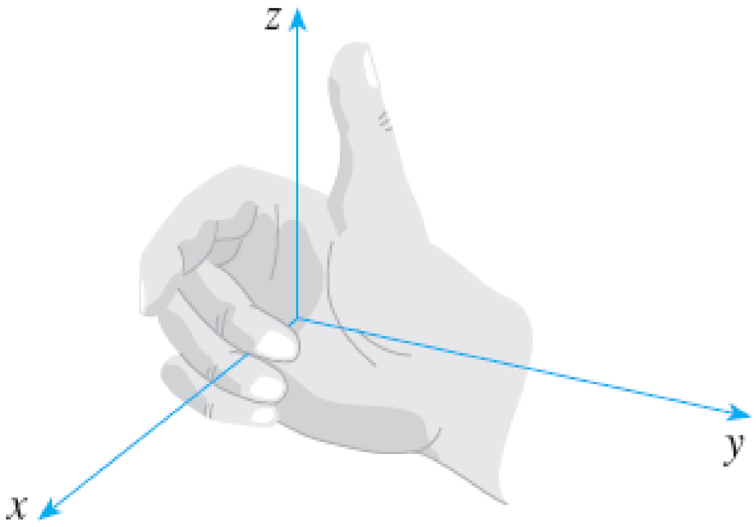


Figure 3. Coordinate axes

LABELLING AXES

Use the right-hand rule when labelling axes.



Right hand:

thumb ≡ z-axis

first finger ≡ x-axis

second finger ≡ y-axis

Figure 4. Right-hand rule

COORDINATE PLANES

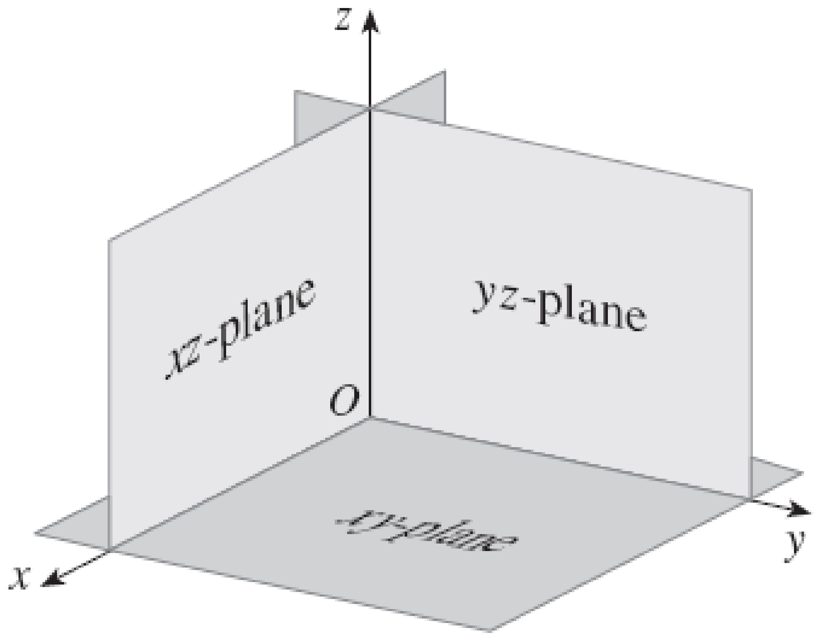


Figure 5. Coordinate planes

COORDINATES OF POINT P

Coordinates of point P written as P(x,y,z).

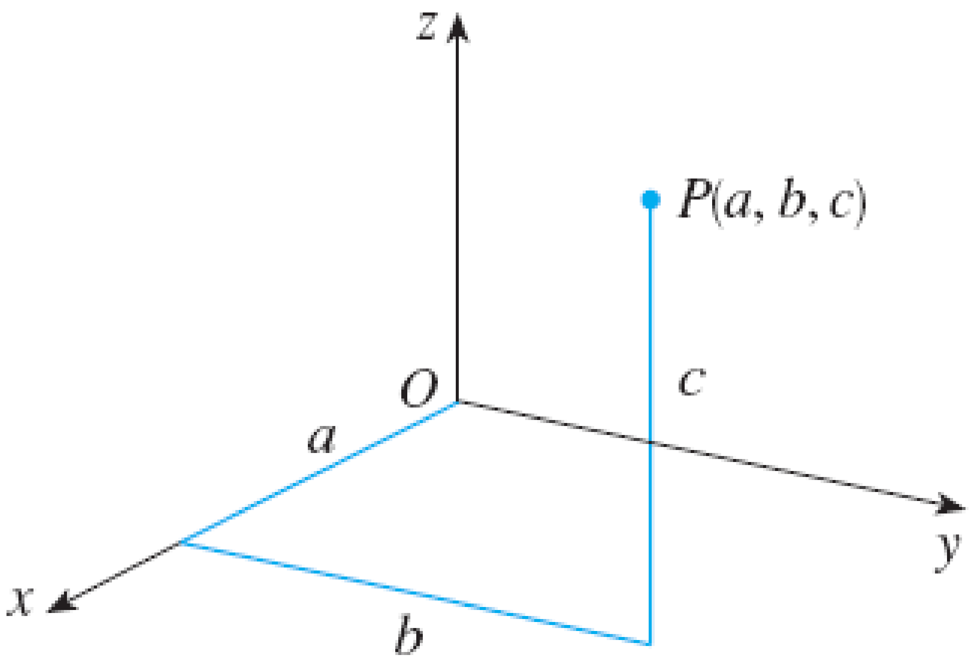
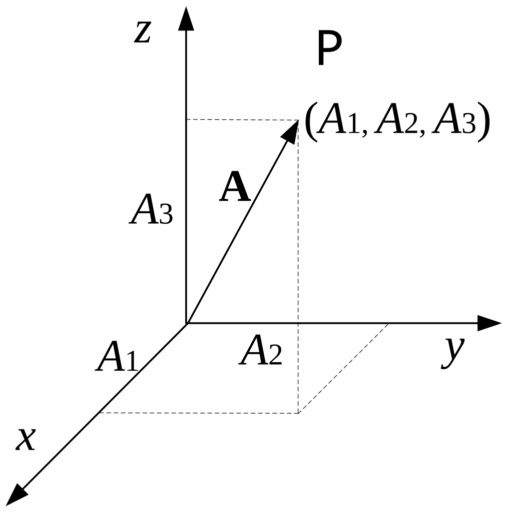


Figure 6. Coordinates of point P.

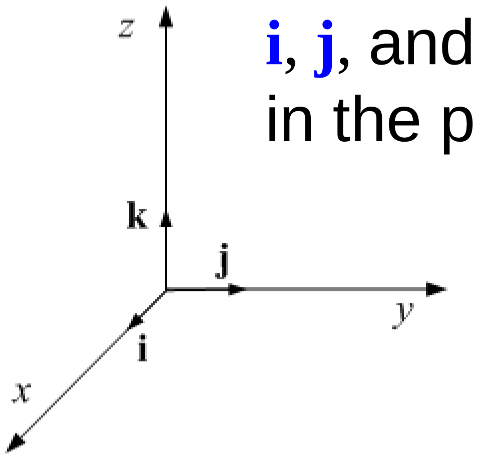
Position vector of a point in space

A point P in Cartesian coordinate system may be expressed as its x,y,z coordinates. The position vector of a point P is the directed distance from the origin O to the point P.



$$\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} = (A_1, A_2, A_3)$$

A_1, A_2 and A_3 are called x, y, and z **component** of vector **A**



i, **j**, and **k** are unit vectors pointing in the positive x, y, and z directions

Position vector of a point in space

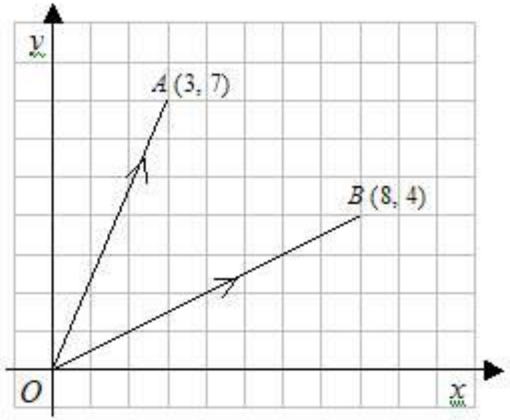
Magnitude of **A**: $|\mathbf{A}| = A = \sqrt{A_1^2 + A_2^2 + A_3^2}$

Direction of **A**: $\hat{\mathbf{a}} = \dfrac{\mathbf{A}}{A} = \dfrac{A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}}{\sqrt{A_1^2 + A_2^2 + A_3^2}}$

Note: $\hat{a}$ is a unit directional vector !

Example 1

Give the position vectors for points A and B, their magnitudes and directions**(c)** The position vector for



Solution

The position vector of a point is nothing more than a vector with the point's coordinates as its components. Therefore,

$$A = 3\hat{i} + 7\hat{j} \qquad\qquad B = 8\hat{i} + 4\hat{j}$$

Solution

Magnitude of **A**:

$$|\mathbf{A}| = A = \sqrt{A_1^{\,2} + A_2^{\,2} + A_3^{\,2}}$$

$$= \sqrt{3^2 + 7^2} = \sqrt{9+49} = \sqrt{58}$$

Direction of **A**:

$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{A} = \frac{A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}}{\sqrt{A_1^{\,2} + A_2^{\,2} + A_3^{\,2}}}$$

$$= \frac{3\hat{i} + 7\hat{j}}{\sqrt{58}} = \frac{3}{\sqrt{58}}\,\hat{i} + \frac{7}{\sqrt{58}}\,\hat{j}$$

Solution

Magnitude of **B**:

$$\left|\vec{B}\right| = B = \sqrt{B_1^2 + B_2^2}$$

$$= \sqrt{8^2 + 4^2} = \sqrt{64+16} = \sqrt{80}$$

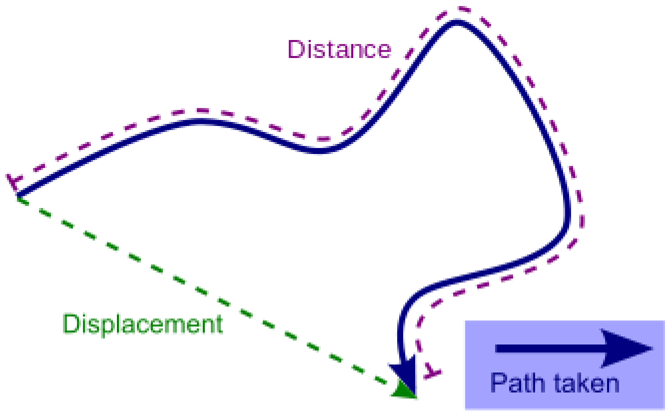
Direction of **B**:

$$\hat{b} = \frac{\vec{B}}{B} = \frac{B_1\hat{i} + B_2\hat{j}}{\sqrt{B_1^2 + B_2^2}}$$

$$= \frac{8\hat{i} + 4\hat{j}}{\sqrt{80}} = \frac{8}{\sqrt{80}}\hat{i} + \frac{4}{\sqrt{80}}\hat{j}$$

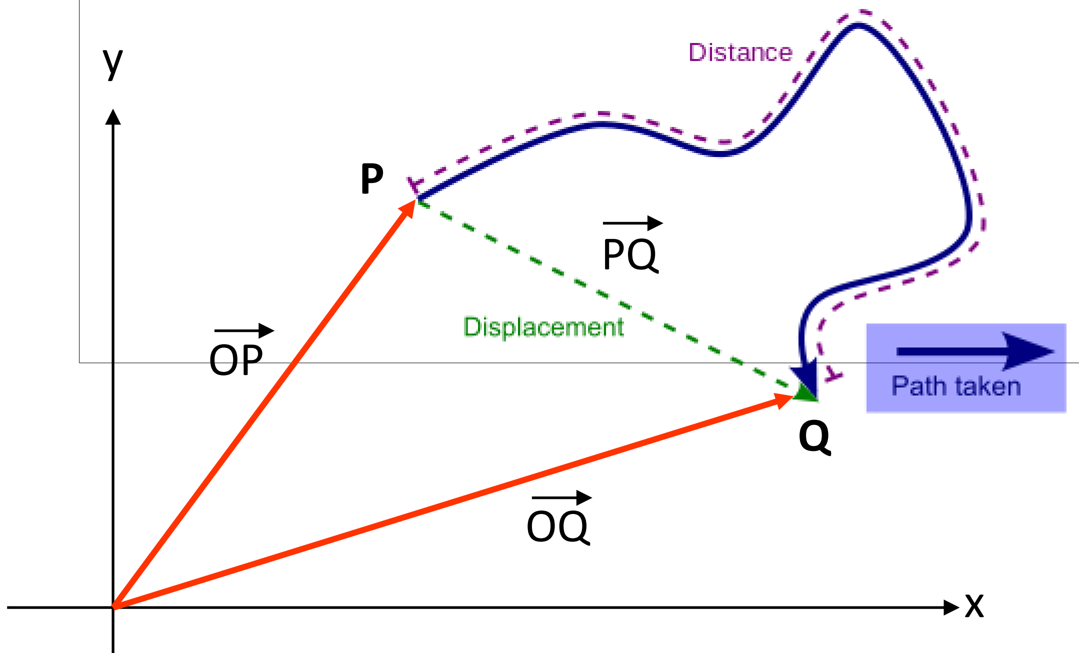
Displacement Vector

A **displacement** is the shortest distance from the initial to the final position of a point P. Thus, it is the length of an imaginary straight path, typically distinct from the path actually travelled by P. A 'displacement vector' represents the length and direction of that imaginary straight path.



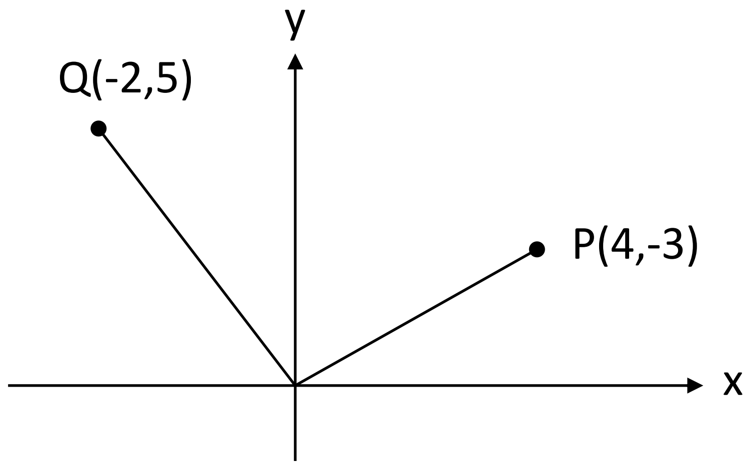
A displacement may be also described as a 'relative position': the final position of a point (**Q**) relative to its initial position (**P**), and a displacement vector can be mathematically defined as the difference between the final and initial position vectors:

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$



Example

The position vector of P is (4,3) and the position vector of Q is (-2,5). Find the displacement vector $\overrightarrow{PQ}$.

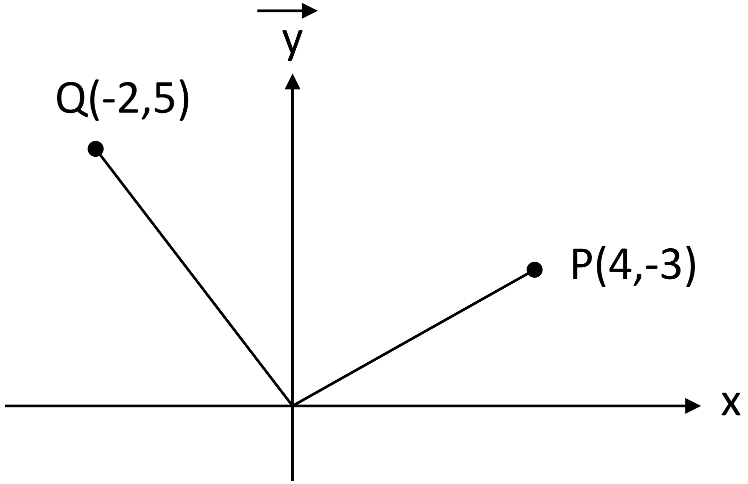


Solution

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$= \begin{pmatrix} -2 \\ 5 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

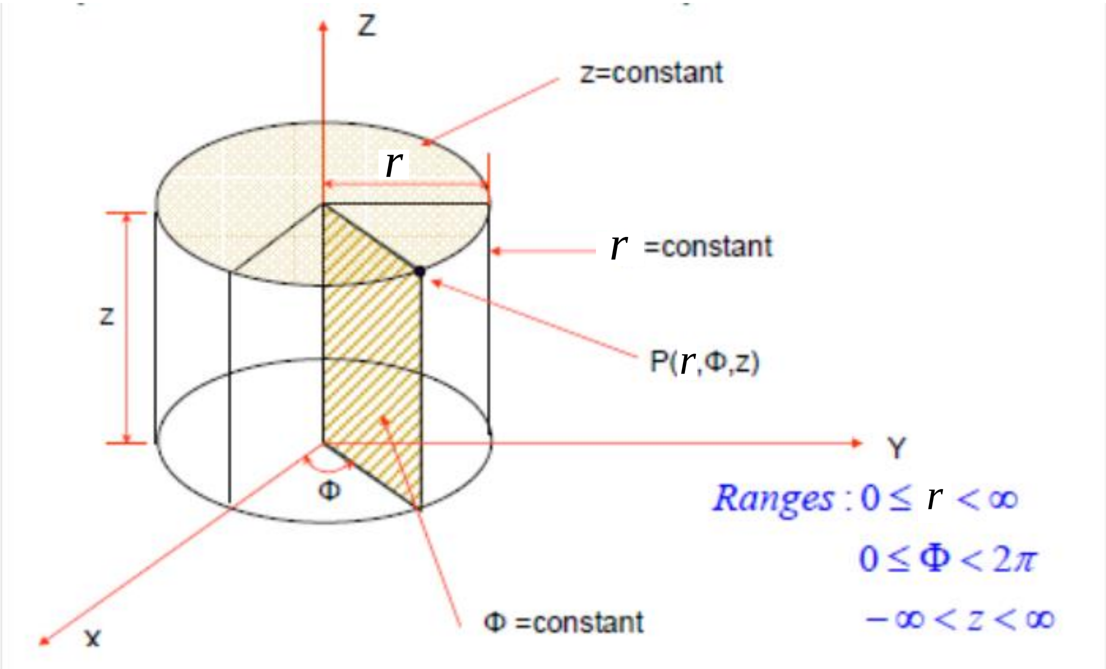
$$= \begin{pmatrix} -6 \\ 2 \end{pmatrix}$$



CYLINDRICAL COORDINATE SYSTEM

i. Any point in space is considered to be at the intersection of three mutually perpendicular surfaces:

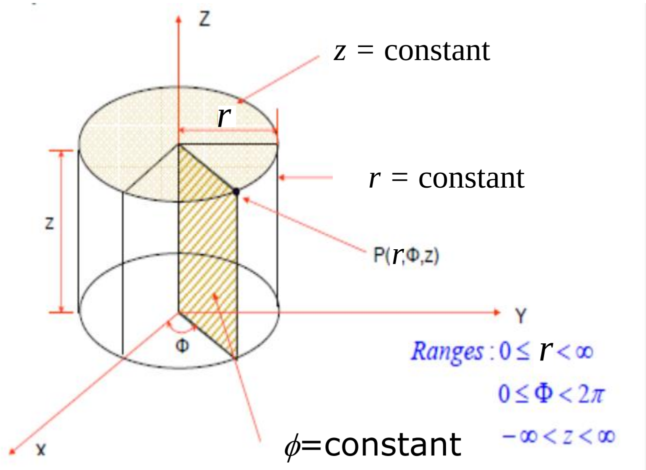
- A circular cylinder (r = constant)
- A vertical plane (ϕ = constant)
- A horizontal plane (z = constant)



Cylindrical Coordinate System

ii. Any point in space is represented by three coordinates P(r,ϕ,z)

- *r* denotes the radius of an imaginary cylinder passing through P, or the radial distance from z axis to the point P
- ϕ Denotes azimuthal angle, measured from x axis to a vertical intersecting plane passing through P
- *z* denotes distance from xy-plane to a horizontal intersecting plane passing through P. It is the same as in rectangular coordinate system



CYLINDRICAL COORDINATE SYSTEMS

iii. A vector in cylindrical coordinate system may be specified using three mutually perpendicular unit vectors $\hat{\mathbf{r}}, \hat{\boldsymbol{\varphi}}, \hat{\mathbf{z}}$

iv. $\hat{\mathbf{r}}, \hat{\boldsymbol{\varphi}}, \hat{\mathbf{z}}$ form a right-handed system because an RH screw when rotated from r to ϕ moves towards z.

v. These unit vector specify directions along r, ϕ, and z axes.

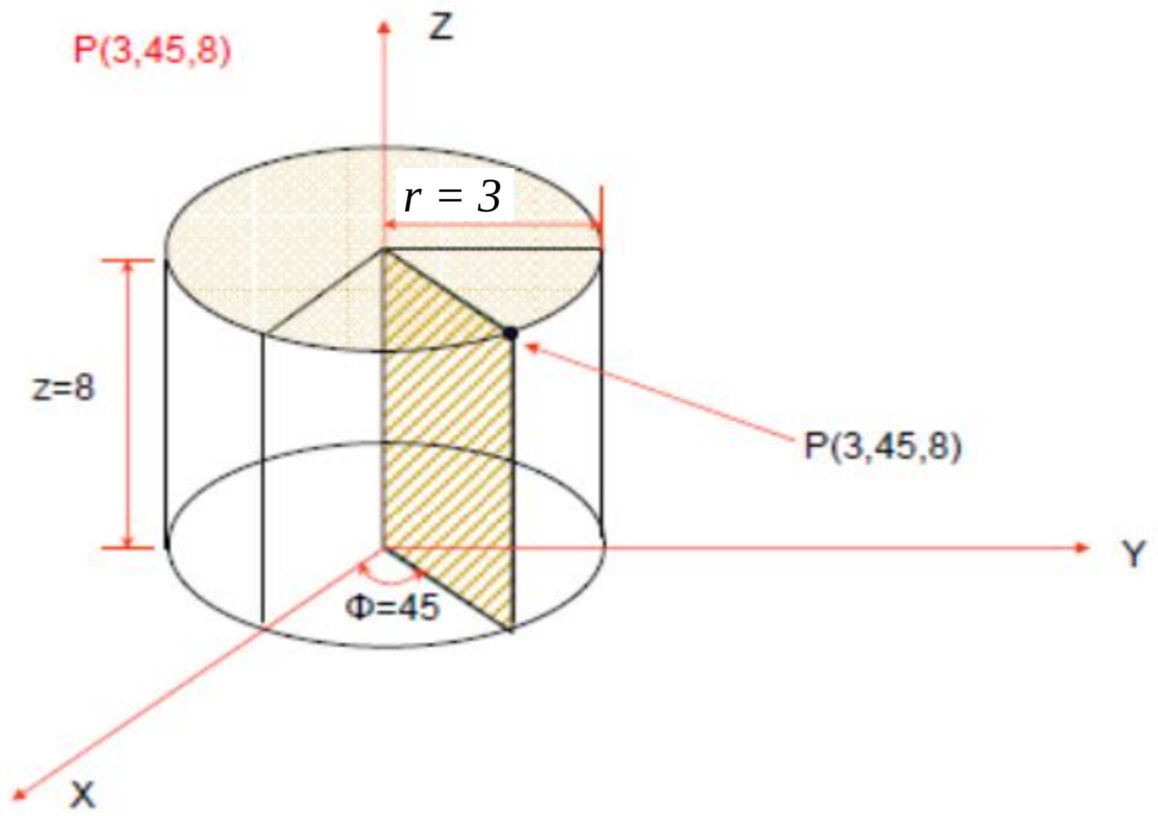
vi. Using these unit vectors any vector ***A*** may be expressed as

$$\mathbf{A} = A_r\hat{\mathbf{r}} + A_\phi\hat{\boldsymbol{\varphi}} + A_z\hat{\mathbf{z}}$$

vii. The magnitude of the vectors is given by

$$|\mathbf{A}| = \sqrt{A_r^2 + A_\phi^2 + A_z^2}$$

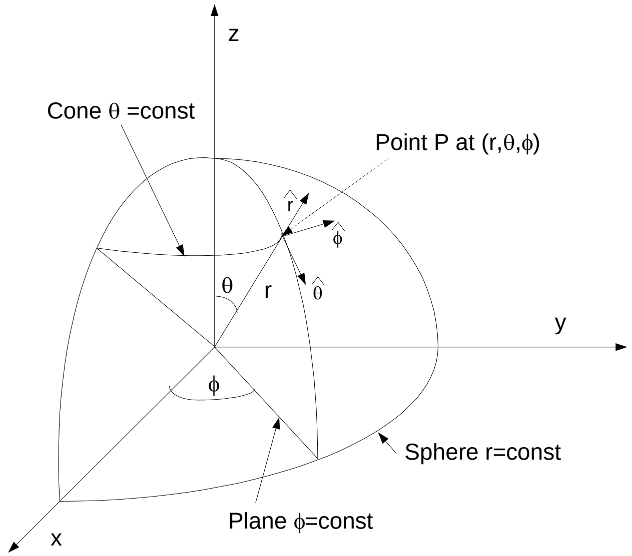
Visualisation of point P(3,45°,8) in the cylindrical coordinate system



SPHERICAL COORDINATE SYSTEM

i. Any point in space is considered to be at the intersection of three mutually perpendicular surfaces:
   - A sphere of radius r from the origin (r = constant)
   - A cone centred around the z axis (θ = constant)
   - A vertical (ϕ = constant)

ii. Any point in spherical coordinate system is considered to be at the intersection of the above three planes.



SPHERICAL COORDINATE SYSTEM

iii. A vector in spherical coordinate system may be specified using three mutually perpendicular unit vectors $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}$

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vi. Using these unit vectors any vector ***A*** may be expressed as

$$\mathbf{A} = A_r \hat{\mathbf{r}} + A_\theta \hat{\boldsymbol{\theta}} + A_\phi \hat{\boldsymbol{\varphi}}$$

vii. The magnitude of the vectors is given by

$$|\mathbf{A}| = \sqrt{A_r^2 + A_\theta^2 + A_\phi^2}$$

Transformation of variables

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z}$$

$$\phi = \tan^{-1} \frac{y}{x}$$

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

DEFINITION OF SCALAR MULTIPLICATION

If c is a scalar and **v** is a vector, then the **scalar multiple** $c\mathbf{v}$ is the vector whose length is c times the length of **v** and whose direction is the same as **v** if $c > 0$ and is opposite to **v** if $c < 0$. If $c = 0$ or $\mathbf{v} = 0$, then $c\mathbf{v} = 0$.

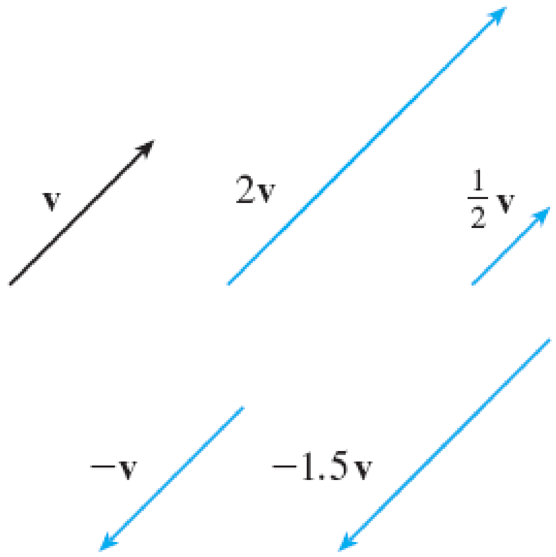

FIGURE 7

Scalar multiples of **v**

Notice that two nonzero vectors are **parallel** if they are scalar multiples of one another. In particular, the vector –v=(-1)v has the same length as v but points in the opposite direction. We call it the **negative** of v.

DEFINITION OF SCALAR MULTIPLICATION

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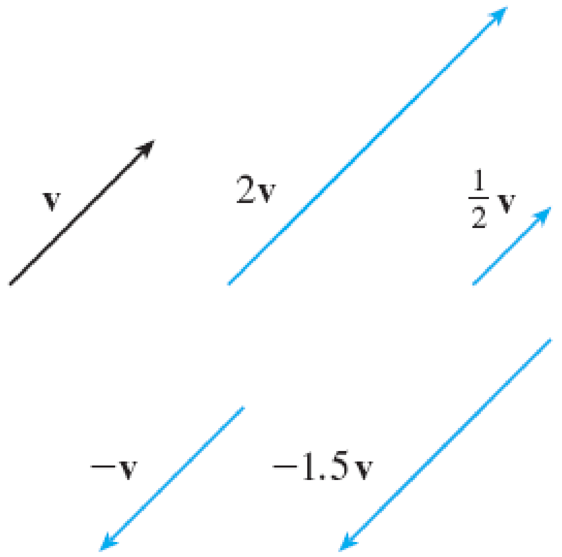

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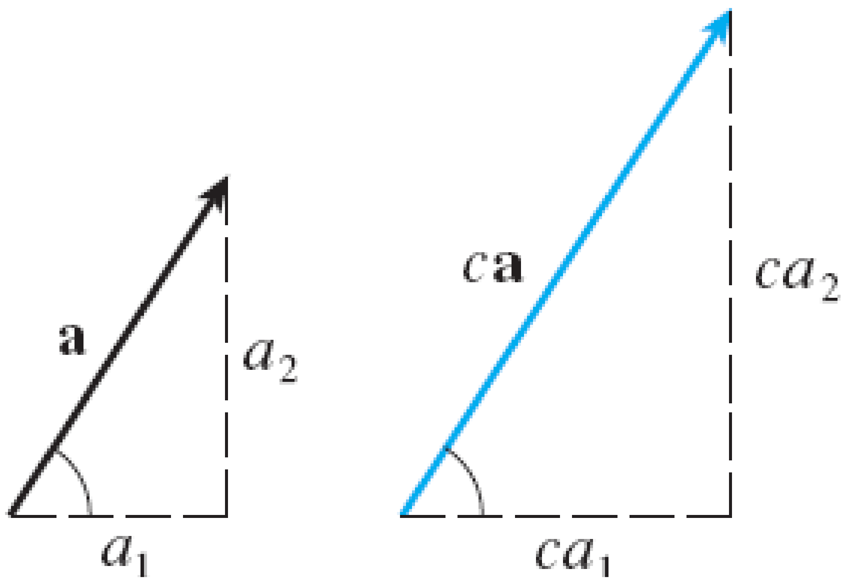
EXAMPLE OF SCALAR MULTIPLICATION

From the similar triangles below we see that the components of c**a** are ca_1 and ca_2. So to multiply a vector by a scalar we multiply each component by that scalar.



EQUIVALENT VECTORS

Displacement vector v , shown in Figure 1, has **initial point A** (the tail) and **terminal point B** (the tip) and we indicate this by writing **v**=$\overrightarrow{AB}$. Notice that the vector **u**=$\overrightarrow{CD}$ has the same length and the same direction as **v** even though it is in a different position. We say that **u** and **v** are **equivalent** (or **equal**) and we write **u**=**v**.

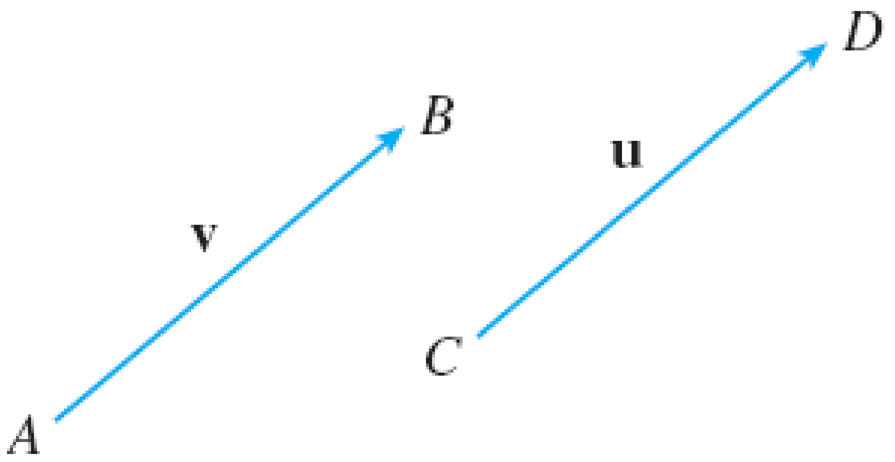
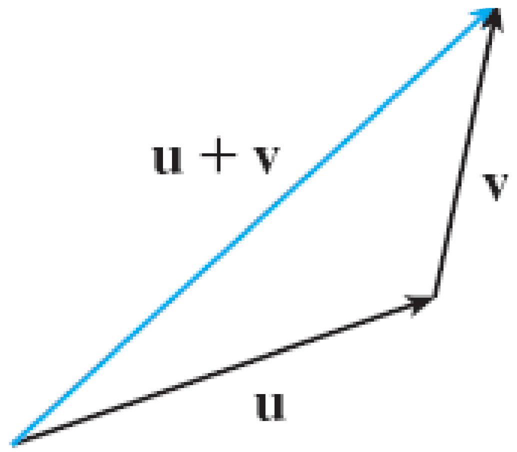


FIGURE 1

Equivalent vectors

DEFINITION OF VECTOR ADDITION

If **u** and **v** are vectors positioned so the initial point of **v** is at the terminal point of **u**, then the sum **u + v** is the vector from the initial point of **u** to the terminal point of **v**.

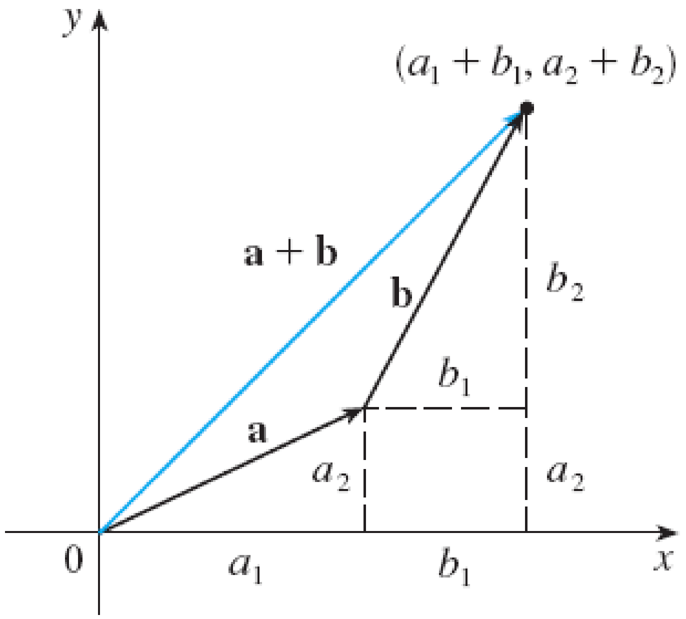


To add algebraic vectors we add their components.

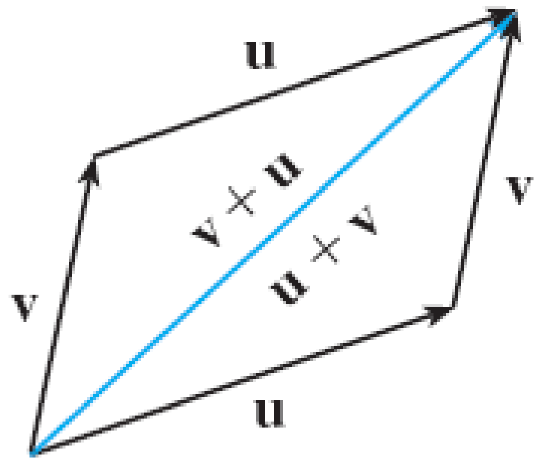
EXAMPLE

If **a**=<a_1,a_2> and **b** =<b_1,b_2>, then the sum is

a + **b**=<a_1+b_1, a_2+b_2>



PARALLELOGRAM LAW

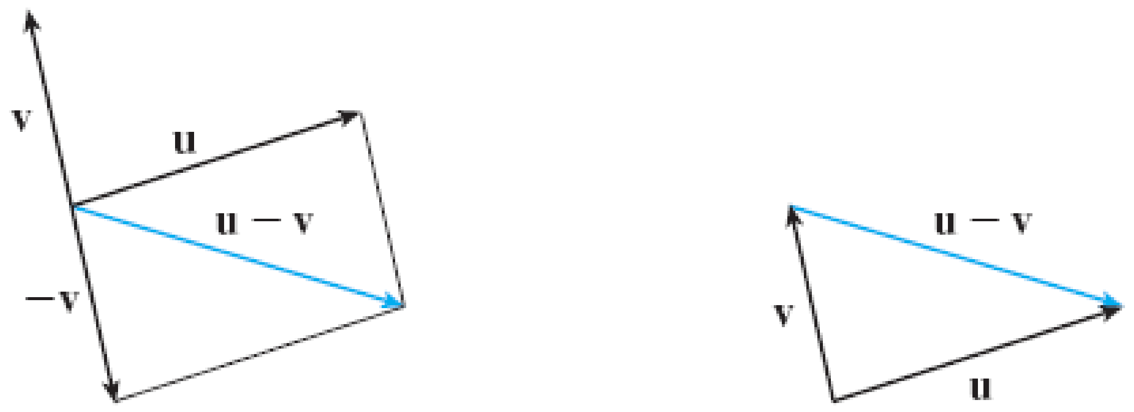


DIFFERENCE OF TWO VECTORS

By the **difference u - v** of two vectors we mean

$$u - v = u + (-v)$$

Drawing **u** – **v**:



(a) (b)

To subtract vectors we subtract components.

EXAMPLE

If $\mathbf{a}$=<a_1,a_2> and $\mathbf{b}$=<b_1,b_2>, then

$$\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

Similarly, for three-dimensional vectors,

$$\mathbf{a} - \mathbf{b} = \langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle$$

$$= \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

PROPERTIES OF VECTORS

If **a**, **b**, and **c** are vectors and d is a scalar, then

1. **a** + **b** = **b** + **a** (Commutative law)

2. **a** + (**b** + **c**) = (**a** + **b**) + **c** (Associative law)

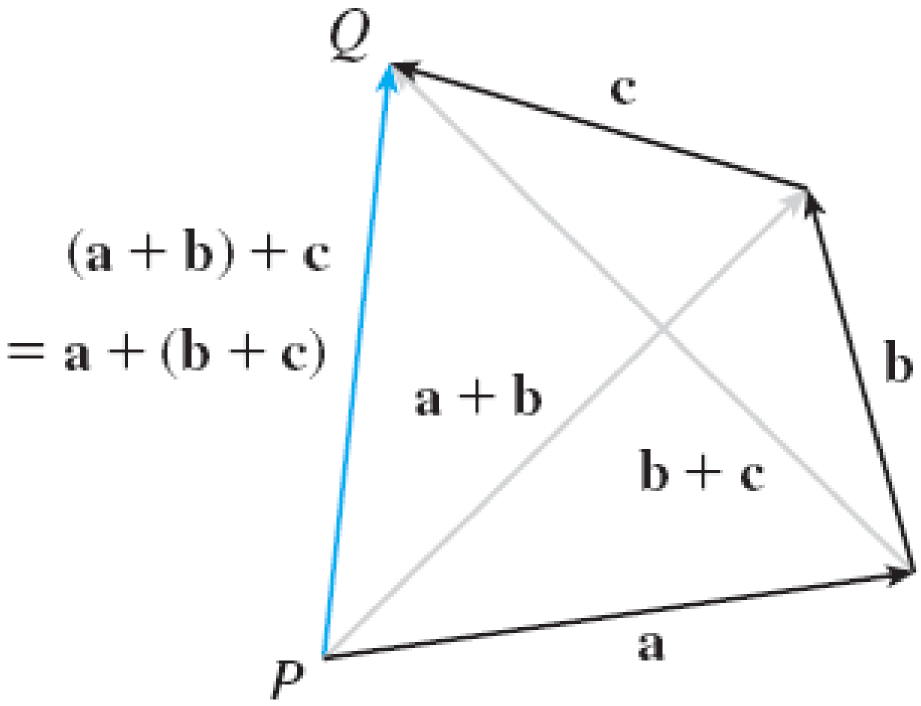
3. **a** + 0 = **a**

4. **a** + (-**a**) = 0

5. d(**a** + **b**) = d**a** + d**b**

6. 1**a** = **a**

GRAPHICAL PROOF



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