Finite Fourier Transforms

Definition:- Finite Fourier Sine Transform

Let $f(t)$ be a function defined on $(0, l)$ and satisfying Dirichlet's conditions $(0, l)$. The finite Fourier sine transform of $f(t)$, $0 < t < l$ is defined as

$$F_s(f(t)) = \int_0^l f(t) \sin \frac{\pi s t}{l} \, dt; \quad s \in N.$$

Definition 2. Inverse Finite Fourier Sine Transform

$f(t)$ is known as inverse finite Fourier sine transform of $F_s(f(t))$ and is defined as

$$F_s^{-1}(F_s(f(t))) = f(t) = \frac{2}{l} \sum_{s=1}^{\infty} F_s(f(t)) \sin \frac{\pi s t}{l}$$

Definition 3. Finite Fourier Cosine Transform

Let $f(t)$ be a function defined on $(0, l)$ and satisfying Dirichlet's conditions on $(0, l)$. Then finite Fourier cosine transform $f(t)$, $0 < t < l$ is defined as

$$F_c(f(t)) = \int_0^l f(t) \cos \frac{\pi s t}{l} \, dt; \quad s \in N.$$

Definition 4. Inverse Finite Fourier Cosine Transform:-

$f(t)$ is known as inverse finite Fourier cosine transform of $F_c(f(t))$ or $F_c(s)$

$$F_c^{-1}(F_c(f(t))) = f(t) = \frac{1}{l} F_c(0) + \frac{2}{l} \sum_{s=1}^{\infty} F_c(s) \cos \frac{\pi s t}{l}$$

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Ex Find finite Fourier sine and cosine transform of $f(t) = t$ on $(0, l)$

Sol $F_s(f(t)) = F_s(t) = \int_0^l f(t) \cdot \sin\left(\frac{\pi s t}{l}\right) dt$

$$\int_0^l t \sin\left(\frac{\pi s t}{l}\right) dt$$

$$\left[t \cdot \frac{-\cos\frac{\pi s t}{l}}{\frac{\pi s}{l}} \right]_0^l - \int_0^l 1 \cdot \frac{\left(-\cos\frac{\pi s t}{l}\right)}{\frac{\pi s}{l}} dt$$

$$-\frac{l}{\pi s}\left(l \cos \pi s - 0\right) + \frac{l}{\pi s}\left(\frac{\sin\frac{\pi s t}{l}}{\frac{\pi s}{l}}\right)_0^l$$

$$\frac{-l^2 (-1)^s}{\pi s} + \frac{l^2}{\pi^2 s^2}\left(\sin \pi s - \sin 0\right)$$

$$\frac{l^2 (-1)^{s+1}}{\pi s} + 0 = \frac{l^2 (-1)^{s+1}}{\pi s}, \quad s \neq 0$$

and $F_c(f(t)) = \int_0^l t \cdot \cos\frac{\pi s t}{l} \, dt$

$$\left[\frac{t \cdot \sin\left(\frac{\pi s t}{l}\right)}{\frac{\pi s}{l}} \right]_0^l - \int_0^l 1 \cdot \frac{\sin\frac{\pi s t}{l}}{\frac{\pi s}{l}} dt$$

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$$= \frac{\ell}{\pi s}(\ell \sin \pi s - 0) - \frac{\ell}{\pi s}\left(\frac{-\cos \pi s t}{\frac{\pi s}{\ell}}\right)_0^t$$

$$0 + \frac{\ell^2}{\pi^2 s^2}(\cos \pi s - \cos 0)$$

$$= \frac{\ell^2}{\pi^2 s^2}\left((-1)^s - 1\right); \quad s \neq 0.$$

$$= \begin{cases} \frac{-2\ell^2}{\pi^2 s^2} & \text{if } s \text{ is odd} \\ 0 & \text{if } s \text{ is even} \end{cases}$$

dt

2 Find finite fourier sine and cosine transform $f(t) = e^{-at} \quad 0 < t < \ell$

Sol $F_s(e^{-at}) = \int_0^\ell e^{-at} \sin\left(\frac{s \pi}{\ell}t\right) dt$

$$\frac{1}{(-a)^2 + \frac{s^2\pi^2}{\ell^2}} e^{-at}\left(-a \sin \frac{s\pi t}{\ell} - \frac{s\pi}{\ell} \cos \frac{s\pi t}{\ell}\right)_0^\ell$$

$$\frac{e^{-a\ell}\ell^2}{a^2\ell^2 + s^2\pi^2}\left(-a \sin s\pi - \frac{s\pi}{\ell} \cos s\pi\right)$$

$$- \frac{e^0 \ell^2}{a^2\ell^2 + s^2\pi^2}\left(-0 - \frac{s\pi}{\ell}\right)$$

$$\frac{e^{-a\ell}\ell^2\left[\frac{-s\pi}{\ell}(-1)^s\right]}{a^2\ell^2 + s^2\pi^2} + \frac{\ell s\pi}{a^2\ell^2 + s^2\pi^2}$$

$$\ell s\pi\left(\frac{+e^{-a\ell}(-1)^{s+1} + 1}{a^2\ell^2 + s^2\pi^2}\right)$$

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$$F_c(e^{-at}) = \int_0^l e^{-at} \cos\left(\frac{st\pi}{l}\right) dt$$

$$\frac{1}{(a)^2 + \frac{s^2\pi^2}{l^2}} e^{-at}\left(-a\cos\frac{s\pi t}{l} + \frac{s\pi}{l}\sin\frac{\pi s t}{l}\right)_0^l$$

$$\frac{l^2 e^{-al}}{a^2l^2 + s^2\pi^2}\left(-a\cos s\pi + \frac{s\pi}{l}\sin\pi s\right) - \frac{l^2 e^0}{a^2l^2 + s^2\pi^2}(-a+0)$$

$$\frac{l^2 e^{-al}\left(-a(-1)^s + 0\right) + al^2}{a^2l^2 + s^2\pi^2}$$

$$\frac{al^2\left(+e^{-al}(-1)^{s+1} + 1\right)}{a^2l^2 + s^2\pi^2}$$

$\cos(\pi)$

3. Find finite Fourier Sine and cosine transform of $f(t) = \cos\alpha t$, when $t \in (0, \pi)$

Sol To find Fourier Sine transform.

case (i) when $\alpha \neq \pm s$

$$F_c(\cos\alpha t) = \int_0^\pi \cos\alpha t \sin st \, dt$$

$$\frac{1}{2}\left(\int_0^\pi \sin(\alpha+s)t - \sin(\alpha-s)t\right)dt = \frac{1}{2}\left[-\frac{\cos(\alpha+s)t}{\alpha+s} + \frac{\cos(\alpha-s)t}{\alpha-s}\right]_0^\pi$$

$$\frac{1}{2}\left(\frac{-\cos(\alpha+s)\pi}{\alpha+s} + \frac{\cos(\alpha-s)\pi}{\alpha-s}\right) - \frac{1}{2}\left(\frac{-1}{\alpha+s} + \frac{1}{\alpha-s}\right)$$

$$\frac{1}{2}\left[\frac{-(\alpha-s)\cos(\alpha+s)\pi + (\alpha+s)\cos(\alpha-s)\pi}{\alpha^2 - s^2}\right] - \frac{1}{2}\left(\frac{-(\alpha-s) + (\alpha+s)}{\alpha^2 - s^2}\right)$$

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$$\frac{1}{2}\left[\frac{\alpha\cos(\alpha-s)\pi-\cos(\alpha+s)\pi+s(\cos(\alpha+s)\pi+\cos(\alpha+s)\pi}{\alpha^2-s^2}\right.$$

$$-\frac{s}{\alpha^2-s^2}$$

$$\frac{1}{2}\left(\frac{\alpha[2\sin\alpha\pi\sin\pi s+s(2\cos\alpha\pi\cos s\pi)}{\alpha^2-s^2}\right]-\frac{s}{\alpha^2-s^2}$$

$$\frac{1}{2}\left(\frac{0+2s\cos\alpha\pi(-1)^s}{\alpha^2-s^2}\right]-\frac{s}{\alpha^2-s^2} \quad \text{if } \alpha\neq\pm s$$

$$=\frac{s}{\alpha^2-s^2}\left((-1)^s\cos\alpha\pi-1\right)$$

Case (ii) when $\alpha = \pm s$

$$F_s(\cos\alpha t) = \int_0^\pi \cos st\sin st\,dt$$

$$\frac{1}{2}\int_0^\pi \sin 2st\,dt$$

$$\frac{1}{2}\left(\frac{-\cos 2st}{2s}\right)_0^\pi$$

$$\frac{-1}{4s}\left(\cos 2s\pi-\cos 0\right)$$

$$\Rightarrow \frac{-1}{4s}(1-1) = 0$$

Further to find cosine transform of $\cos\alpha t$

Case (i) when $\alpha\neq\pm s$

$$F_c(\cos\alpha t) = \int_0^\pi \cos\alpha t\cos st\,dt$$

$$\frac{1}{2}\int_0^\pi \cos(\alpha+s)t+\cos(\alpha-s)t\,)dt$$

$$\frac{1}{2}\left[\frac{\sin(\alpha+s)t}{\alpha+s}+\frac{\sin(\alpha-s)t}{\alpha-s}\right]_0^\pi$$

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$$\frac{1}{2}\left(\frac{\sin(\alpha+s)\pi}{\alpha+s} + \frac{\sin(\alpha-s)\pi}{\alpha-s}\right) - (\sin 0 - \sin 0)$$

$$\frac{1}{2}\left(\frac{(\alpha-s)\sin(\alpha+s)\pi + (\alpha+s)\sin(\alpha-s)\pi}{(\alpha-s)(\alpha+s)}\right.$$

$$\frac{1}{2}\left(\frac{\alpha[2\cos s\pi \sin\alpha\pi] - s[2\sin s\pi \cos\alpha\pi]}{\alpha^2-s^2}\right]$$

$$\frac{1}{2}\left(\frac{2\alpha \sin\alpha\pi\,(-1)^s - 0)}{\alpha^2-s^2}\right.$$

$$\frac{(-1)^s\alpha(\sin\alpha\pi)}{\alpha^2-s^2} \quad ; \quad \alpha \neq \pm s$$

when $\alpha = \pm s$

$$F_c(\cos\alpha t) = \int_0^\pi \cos st\,\cos st\,dt = \int_0^\pi \cos^2 st\,dt$$

$$\int_0^\pi \left(\frac{1+\cos 2st}{2}\right)dt$$

$$\frac{1}{2}\left(t + \frac{\sin 2st}{2s}\right)_0^\pi = \pi/2$$