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Applications of Fourier Transform

Results :-

(i) $F_s\left(\frac{\partial^2 v}{\partial x^2}\right) = s\,v(0,t) - s^2 \bar{v}_s(s,t)$

where $\bar{v}_s(s,t)$ is fourier sine transform of $v(x,t)$ w.r.t x

(ii) $F_c\left(\frac{\partial^2 v}{\partial x^2}\right) = -\left(\frac{\partial v}{\partial x}\right)_{x=0} - s^2 \bar{v}_c(s,t)$

where $\bar{v}_c(s,t)$ is fourier cosine transform of $v(x,t)$ w.r.t x.

(iii) $F\left(\frac{\partial^2 v}{\partial x^2}\right) = -s^2 F(v)$

where $F(v)$ is fourier transform of v w.r.t x.

Note :- If $v(x,t)$ at $x=0$ is given, take fourier sine transform and if $v_x(x,t)$ at $x=0$ is given, take fourier cosine transform.

Ex Solve $\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}$, $x>0$, $t>0$

(i) $v(0,t)=0$ i.e at $x=0$, $v=0$

(ii) $v(x,0) = \begin{cases} 1, & 0<x<1 \\ 0, & x>1 \end{cases}$

(iii) $v(x,t)$ is bounded

Sol Given P.D.E is $\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}$ — (i)

As $v=0$, $x=0$

Take fourier sine transform both sides of (i)

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$$\int_0^\infty \frac{\partial v}{\partial t} \sin sx\, dx = \int_0^\infty \frac{\partial^2 v}{\partial x^2} \sin sx\, dx$$

$$\Rightarrow \frac{d}{dt}\left[\int_0^\infty v \sin sx\, dx\right] = F_s\left(\frac{\partial^2 v}{\partial x^2}\right)$$

$$\Rightarrow \frac{d}{dt}\bar{v}_s = s(0) - s^2(\bar{v}_s(s,t)), \quad \bar{v}_s \Rightarrow$$ Fourier sine trans form of v.

$$\Rightarrow \frac{d\bar{v}_s}{dt} = -s^2 \bar{v}_s(s,t)$$

$$\Rightarrow \frac{d\bar{v}_s}{dt} + s^2 \bar{v}_s = 0$$

$$\Rightarrow \frac{d\bar{v}_s}{dt\,\bar{v}_s} + \frac{s^2 \cancel{\bar{v}_s}}{\cancel{\bar{v}_s}} = 0$$

$$\Rightarrow \frac{d\bar{v}_s}{dt\,\bar{v}_s} + s^2 = 0$$

$$\left[\because \int_0^\infty \frac{\partial^2 v}{\partial x^2} \sin sx\, dx = F_s\left(\frac{\partial^2 v}{\partial x^2}\right)\right.$$

$$\left.\Rightarrow F_s\left(\frac{\partial^2 v}{\partial x^2}\right) = s\, v(0,t) - s^2 \bar{v}_s(s,t)\right]$$

$$\Rightarrow \frac{d\bar{v}_s}{\bar{v}_s} = -s^2 \times dt$$

integrating

$$\int \frac{d\bar{v}_s}{\bar{v}_s} = -\int s^2 dt$$

$$\Rightarrow \log \bar{v}_s = -s^2 t + c', \quad c' \Rightarrow \text{constant}$$

$$\Rightarrow \bar{v}_s = e^{-s^2 t + c'}$$

$$\Rightarrow \bar{v}_s = e^{c'} \cdot e^{-s^2 t}$$

$$\Rightarrow \bar{v}_s = c\, e^{-s^2 t} \qquad [e^{c'} = c]$$

put $t = 0$

$$\bar{v}_s = c\, e^{-s^2(0)}$$

$$\bar{v}_s = c$$

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$\Rightarrow C = \bar{v}_s(s,0) = \int_0^\infty v(x,0)\sin sx\,dx$

$\Rightarrow \int_0^1 v(x,0)\sin sx\,dx + \int_1^\infty v(x,0)\sin sx\,dx$

$\int_0^1 1\cdot \sin sx\,dx + \int_1^\infty 0\cdot \sin sx\,dx$

$\Rightarrow -\left(\frac{\cos sx}{s}\right)_0^1 + 0 = \frac{-1}{s}(\cos s - 1)$

$\Rightarrow \frac{1-\cos s}{s}$

Hence $\bar{v}_s = +\left(\frac{1-\cos s}{s}\right) e^{-s^2 t}$

Further taking Inverse Fourier sine transform

$v(x,t) = \frac{2}{\pi}\int_0^\infty \frac{1-\cos s}{s} e^{-s^2 t} \sin sx\,ds$

Ex 2 Solve $\frac{\partial v}{\partial t} = c\frac{\partial^2 v}{\partial x^2}$; $x>0$ with conditions

(i) $v_x = -k$ (A positive constant k) at $x=0, t>0$.

(ii) $v=0$ at $t=0$, $x>0$

Given P.D.E is $\frac{\partial v}{\partial t} = c\frac{\partial^2 v}{\partial x^2}$ —— (I)

As $v_x = -k$ is given at $x=0$
we take fourier cosine transform
on both sides of (i)

$\int_0^\infty \frac{\partial v}{\partial t}\cos sx\,dx = c\int_0^\infty \frac{\partial^2 v}{\partial x^2}\cos sx\,dx$.

$$\Rightarrow \frac{d}{dt}\left(\int_0^\infty v\cos sx\,dx\right) = d\,F_c\left(\frac{\partial^2 v}{\partial x^2}\right)$$

$$= d\left(-(v_x)_{x=0}\right) - s^2\bar{v}_c$$

$$\Rightarrow \frac{d}{dt}\bar{v}_c = d(-k) - ds^2\bar{v}_c$$

$$\Rightarrow \frac{d\bar{v}_c}{\bar{v}_c} + ds^2\bar{v}_c = dk$$

$$\text{I.F} \quad e^{\int ds^2 dt} = e^{ds^2 t}$$

$\therefore$ the general solution of (ii) is

$$e^{ds^2 t}\bar{v}_c = \int dk\,e^{ds^2 t}\,dt + A$$

$$\Rightarrow e^{ds^2 t}\bar{v}_c = \frac{dk\,e^{ds^2 t}}{ds^2} + A$$

$$\Rightarrow \bar{v}_c = \frac{k}{s^2} + Ae^{-ds^2 t}$$

Also given $v(x,0)=0$

Taking fourier cosine transform,
we get $\bar{v}_c(s,0)=0$

Put $t=0$ in (iii) we get $0 = \frac{k}{s^2} + A$

$$\Rightarrow A = \frac{-k}{s^2}$$

$$\therefore \bar{v}_c = \frac{k}{s^2} - \frac{k}{s^2}e^{-ds^2 t} = \frac{k}{s^2}\left(1 - e^{-ds^2 t}\right.$$

Take inverse fourier cosine transform, we get

$$v(x,t) = \frac{2}{\pi}\int_0^\infty \frac{k}{s^2}\left(1-e^{-a s^2 t}\right)\cos sx\, ds$$

Applications of Finite Fourier Transforms

Result:- For $0 \le x \le l$

(i) $F_s\left(\frac{\partial v}{\partial x}\right) = -\frac{s\pi}{l} F_c(v)$

(ii) $F_c\left(\frac{\partial v}{\partial x}\right) = \frac{s\pi}{l} F_s(v) - v(0,t) + (-1)^s v(l,t)$

(iii) $F_s\left(\frac{\partial^2 v}{\partial x^2}\right) = -\frac{s^2\pi^2}{l^2} F_s(v) + \frac{s\pi}{l}\left(v(0,t) - (-1)^s v(l,t)\right)$

(iv) $F_c\left(\frac{\partial^2 v}{\partial x^2}\right) = -\frac{s^2\pi^2}{l^2} F_c(v) + (-1)^p \frac{\partial v(l,t)}{\partial x} - \frac{\partial v(0,t)}{\partial x}$

Ex 1. Solve $\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}$ using finite fourier transform if $v(0,t)=0$ and $v(4,t)=0$ and $v(x,0) = 4x$ where $x \in (0,4)$; $t>0$

Solution Given P.D.E is $\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}$

Subject to $v(0,t) = 0$, $v(4,t)=0$

and $v(x,0) = 4x$ where $x \in (0,4)$; $t>0$

As $v(0,t)$ and $v(4,t)$ are given, so we should take finite fourier

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Sine transform on both sides of (i)
(Here $l = 4$)

$$\int_0^4 \frac{\partial v}{\partial t} \sin \frac{s\pi x}{4} dx = \int_0^4 \frac{\partial^2 v}{\partial x^2} \sin \frac{s\pi x}{4} dx$$

$$\Rightarrow \frac{d}{dt}\left(\int_0^4 v \sin \frac{s\pi x}{4} dx\right) = \left(\frac{\partial v}{\partial x} \sin \frac{s\pi x}{4}\right)_0^4 - \int_0^4 \frac{\partial v}{\partial x}\left(\frac{s\pi}{4} \cos \frac{s\pi x}{4}\right) dx$$

$$\Rightarrow \frac{d\bar{v}_s}{dt} = 0 - \frac{s\pi}{4} \int_0^4 \frac{\partial v}{\partial x} \cos \frac{s\pi x}{4} dx$$

$$= -\frac{s\pi}{4}\left[\left(v \cos \frac{s\pi x}{4}\right)_0^4 - \int_0^4 v\left(-\frac{s\pi}{4} \sin \frac{s\pi x}{4}\right) dx\right]$$

$$= -\frac{s\pi}{4}\left\{v(4,t) \cos s\pi - v(0,t) \cos 0\right\} + \frac{s\pi}{4} \int_0^4 v \sin \frac{s\pi x}{4} dx$$

$$= -\frac{s\pi}{4}\left(0 - 0 + \frac{s\pi}{4} \int_0^4 v \sin \frac{s\pi x}{4} dx\right)$$

$$\Rightarrow -\frac{s^2\pi^2}{16} \bar{v}_s$$

$$\Rightarrow \frac{d\bar{v}_s}{dt} = \frac{-s^2\pi^2 \bar{v}_s}{16}$$

$$\Rightarrow \frac{d\bar{v}_s}{\bar{v}_s} = \frac{-s^2\pi^2}{16} dt$$

Integrating $\log \bar{v}_s = -\frac{s^2\pi^2}{16} t + c'$

$$\Rightarrow \bar{v}_s = e^{-\frac{s^2\pi^2}{16}t + c'} = e^{c'} e^{-\frac{s^2\pi^2}{16}t}$$

$$\Rightarrow \bar{v}_s(s,t) = c\, e^{-\frac{s^2\pi^2}{16}t}$$

Further $t=0$ we get

$$\bar{v}_s(s,0) = c e^0 = c$$

$$\Rightarrow c = \bar{v}_s(s,0) = \int_0^4 v(x,0) \sin\frac{s\pi x}{4}\, dx$$

$$\int_0^4 4x \sin\frac{s\pi x}{4}\, dx$$

$$\Rightarrow c = 4\left[x\left(\frac{-\cos\frac{s\pi x}{4}}{\frac{s\pi}{4}}\right)_0^4 - \int 1\cdot\left(\frac{-\cos\frac{s\pi x}{4}}{\frac{s\pi}{4}}\right) dx \right]$$

$$\Rightarrow c = 4\left[\frac{-4\times 4}{s\pi}\cos s\pi + 0 + \frac{4}{s\pi}\left(\frac{\sin\frac{s\pi x}{4}}{\frac{s\pi}{4}}\right)_0^4\right]$$

$$\Rightarrow c = 4\left[\frac{-16}{s\pi}(-1)^s + \frac{16}{s^2\pi^2}\left(\sin s\pi - \sin 0\right)\right]$$

$$\Rightarrow c = \frac{64\,(-1)^{s+1}}{s\pi} + 0$$

$$\therefore \bar{v}_s(s,t) = \frac{(-1)^{s+1}\, 64}{s\pi}\, e^{-\frac{s^2\pi^2}{16}t}$$